

Cardinality

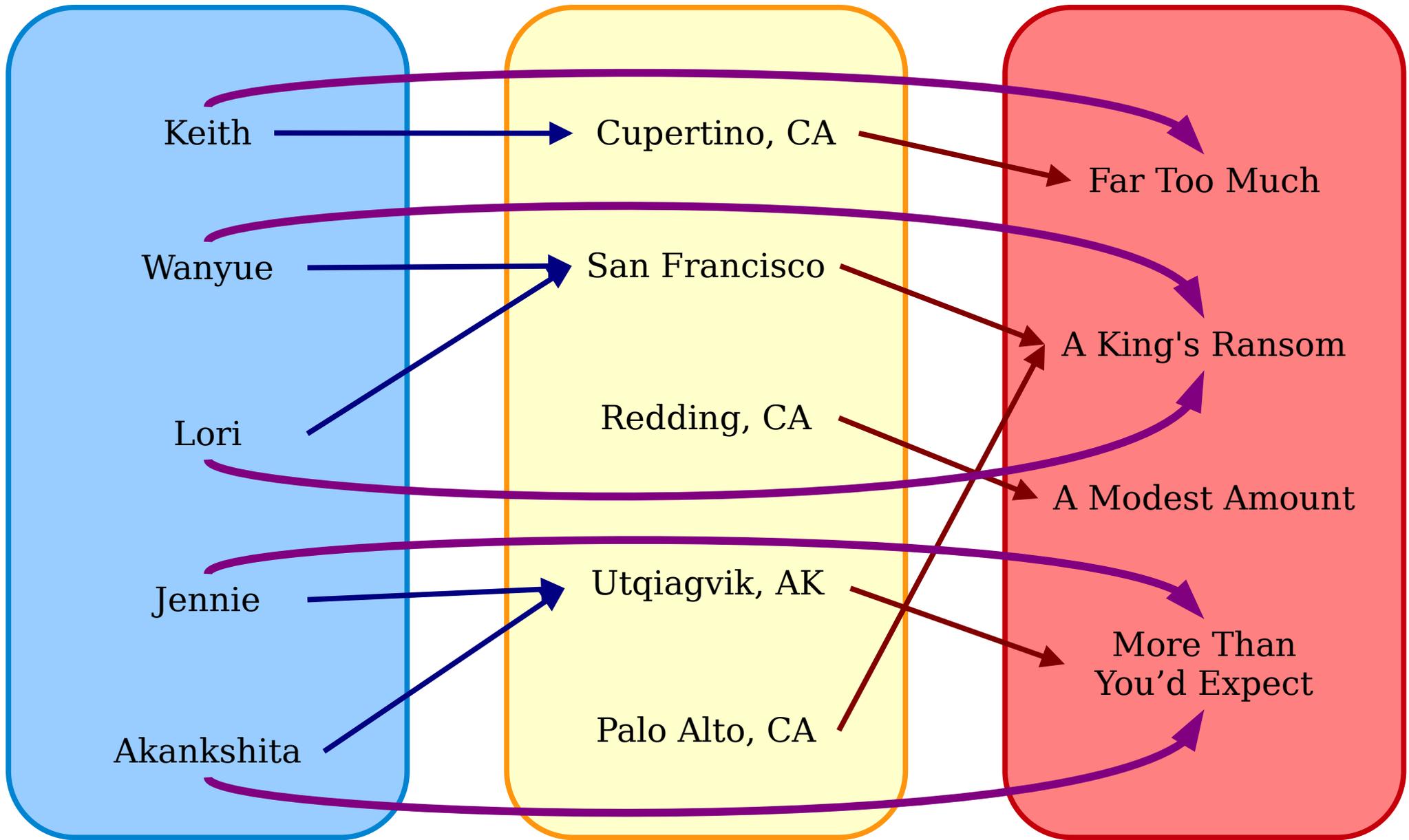
Outline for Today

- ***Function Composition***
 - Combining functions together.
- ***Bijections***
 - A key and important class of functions.
- ***Cardinality, Formally***
 - What does it mean for two sets to have the same size?
- ***Cantor's Theorem, Formally***
 - Proving that infinity is not infinity is not infinity.

Function Composition

f : People → Places

g : Places → Prices



People

Places

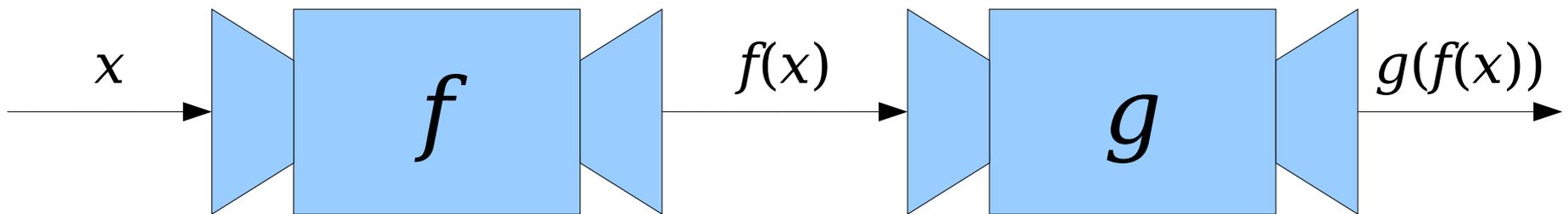
Prices

h : People → Prices

$h(x) = g(f(x))$

Function Composition

- Suppose that we have two functions $f : A \rightarrow B$ and $g : B \rightarrow C$.
- Notice that the codomain of f is the domain of g . This means that we can use outputs from f as inputs to g .



Function Composition

- Suppose that we have two functions $f : A \rightarrow B$ and $g : B \rightarrow C$.
- The **composition of f and g** , denoted $g \circ f$, is a function where
 - $g \circ f : A \rightarrow C$, and
 - $(g \circ f)(x) = g(f(x))$.
- A few things to notice:
 - The domain of $g \circ f$ is the domain of f . Its codomain is the codomain of g .
 - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

The name of the function is $g \circ f$.
When we apply it to an input x ,
we write $(g \circ f)(x)$. I don't know
why, but that's what we do.

Properties of Composition

Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is an injection.

Organizing Our Thoughts

Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is an injection.

What We're Assuming

$f : A \rightarrow B$ is an injection.

$\forall x \in A. \forall y \in A. (x \neq y \rightarrow$
 $f(x) \neq f(y))$

$g : B \rightarrow C$ is an injection.

$\forall x \in B. \forall y \in B. (x \neq y \rightarrow$
 $g(x) \neq g(y))$

We're *assuming* these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow$
 $(g \circ f)(a_1) \neq (g \circ f)(a_2))$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

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Now we're looking at an implication. Let's *assume* the antecedent and *prove* the consequent.

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Let's write this out separately and simplify things a bit.

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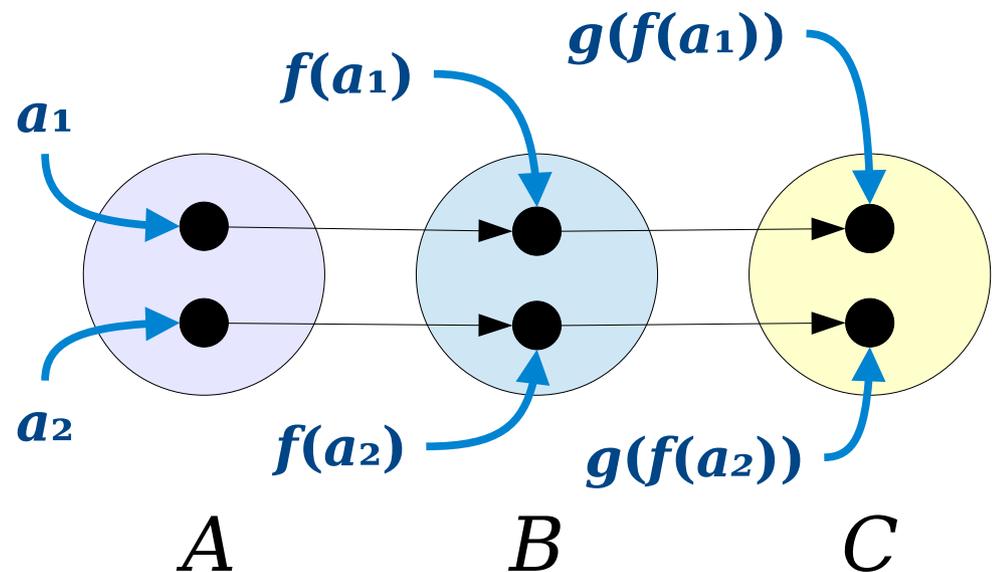
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What We Need to Prove

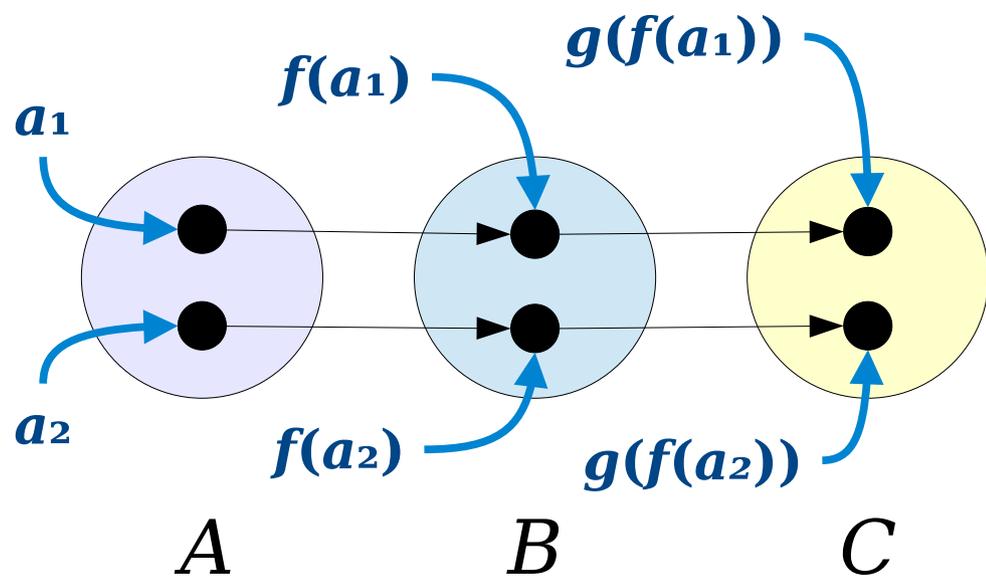
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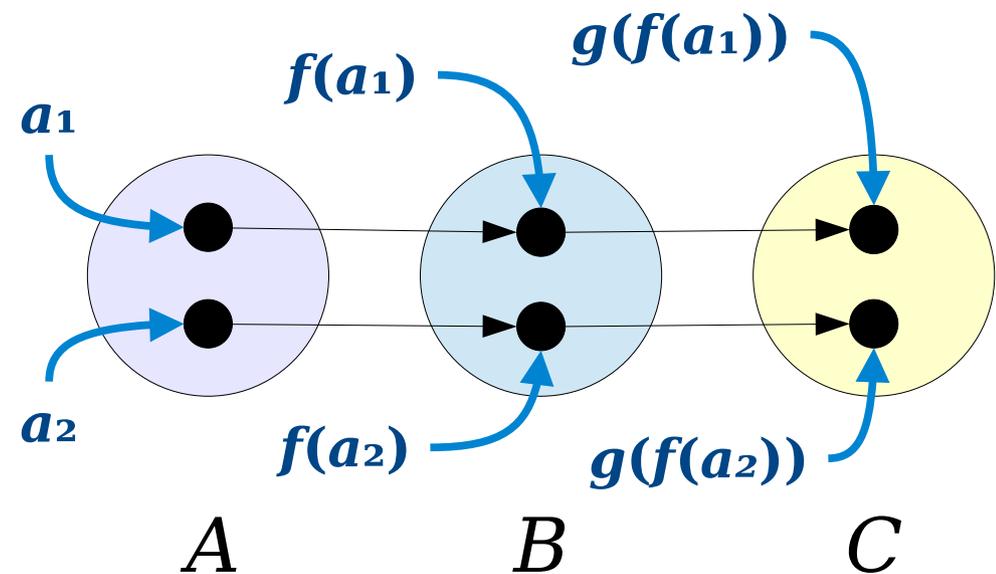


Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is also an injection.



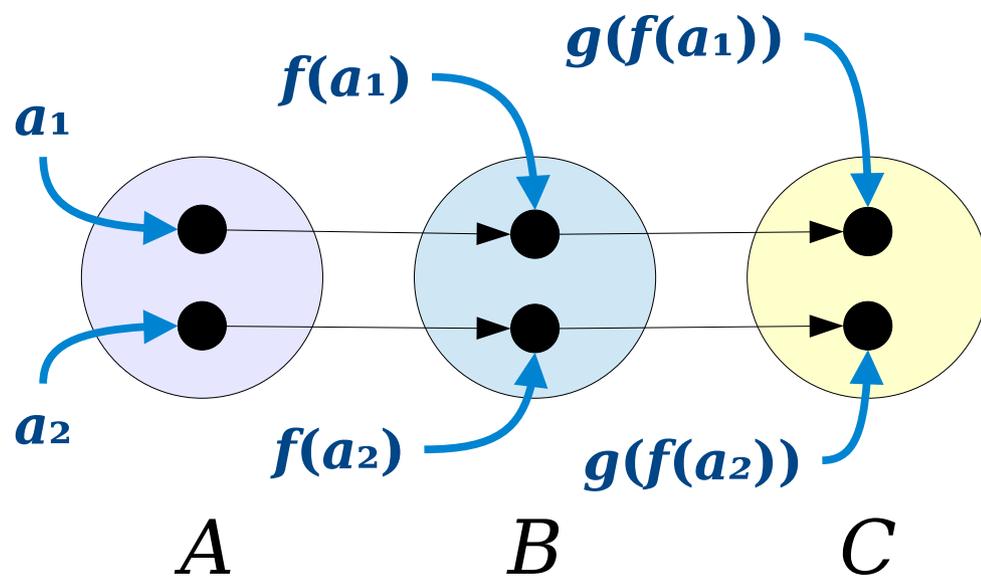
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Proof:



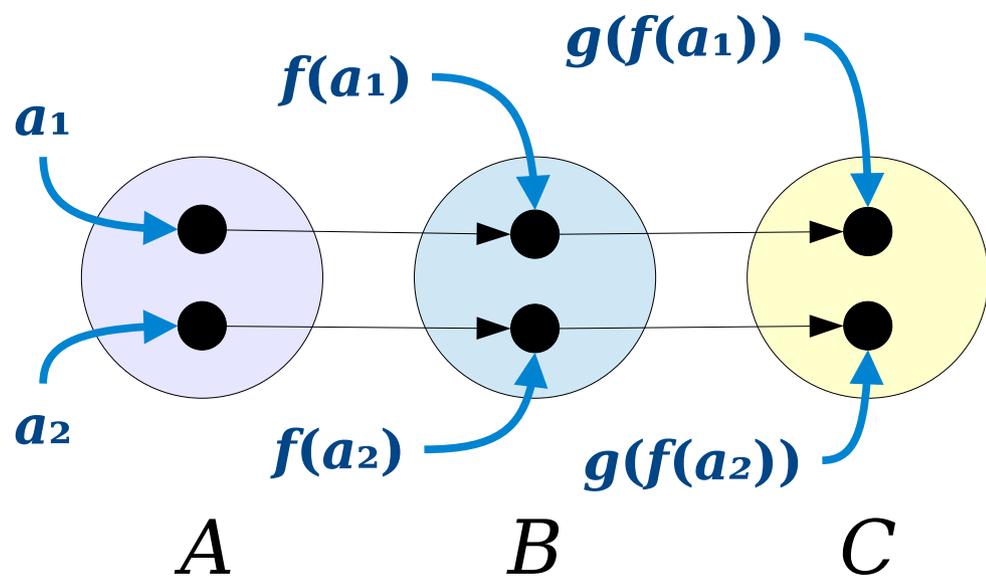
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Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary injections.



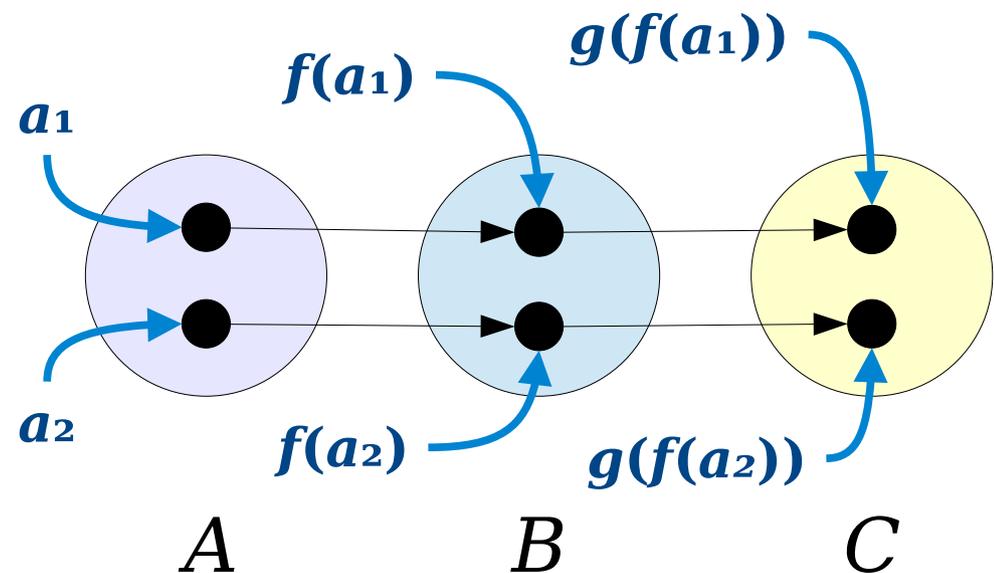
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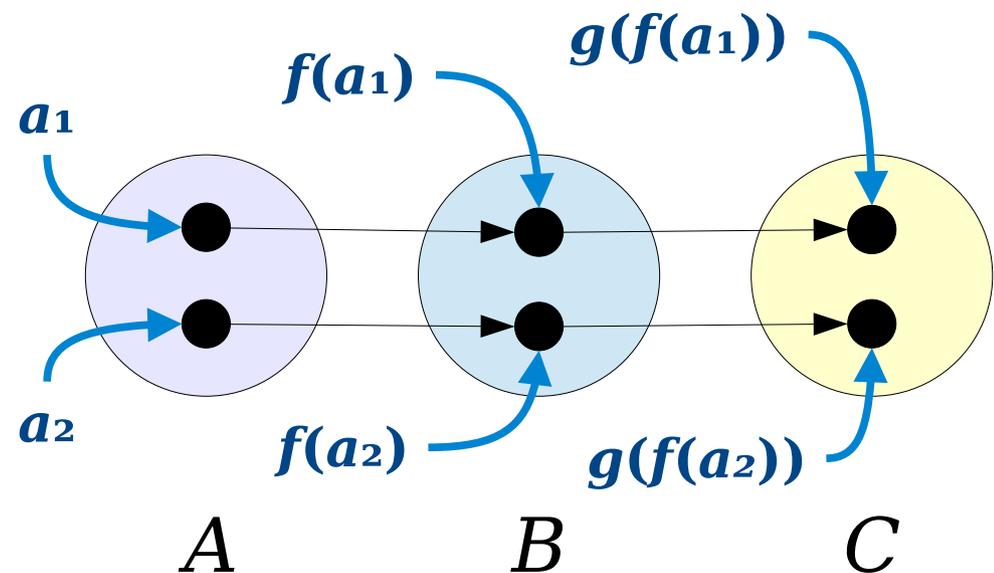
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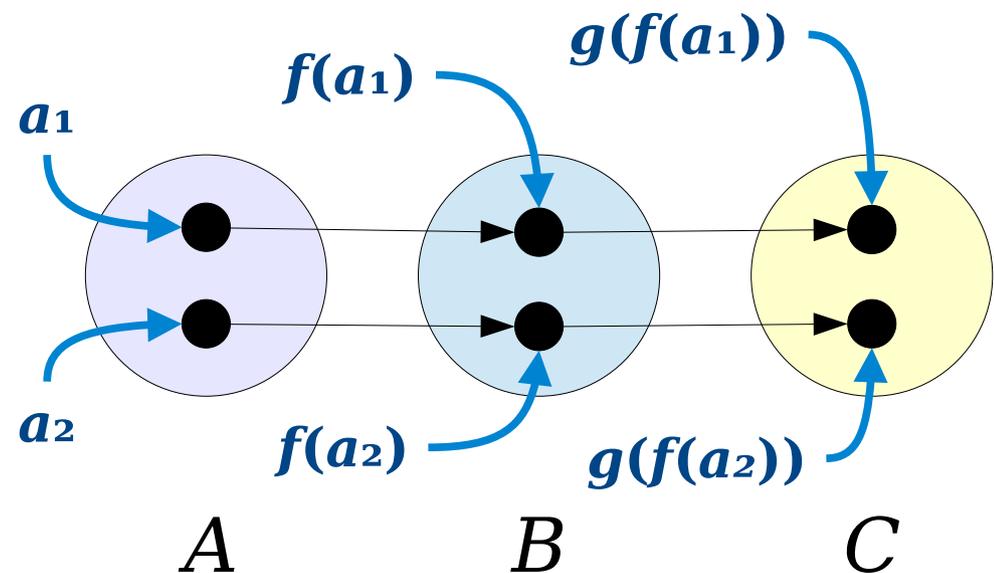
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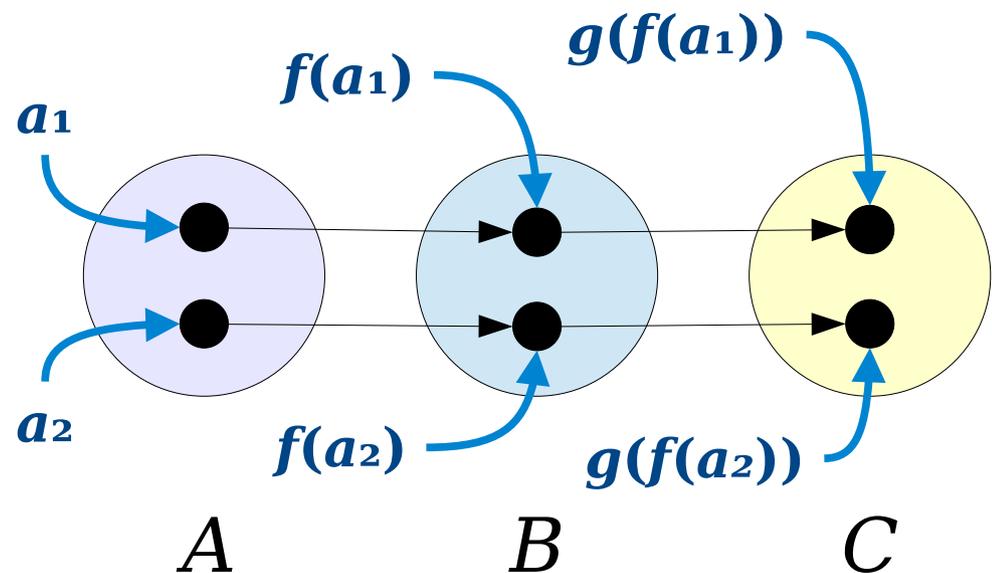
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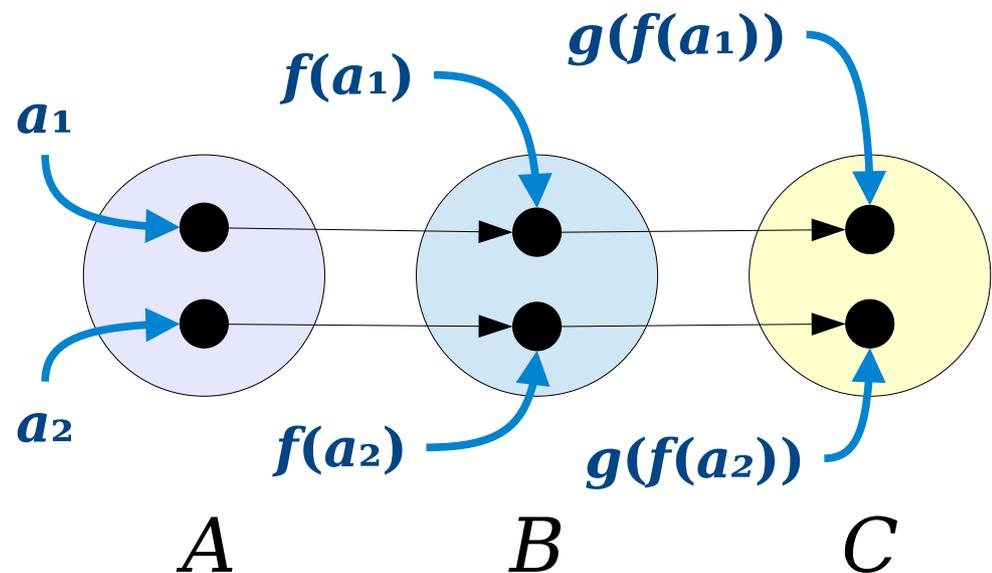
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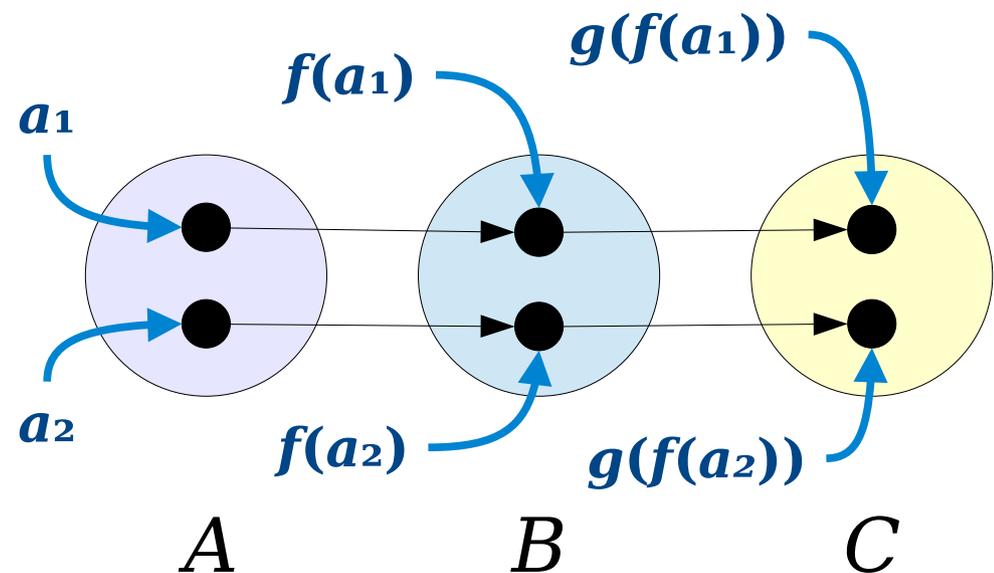
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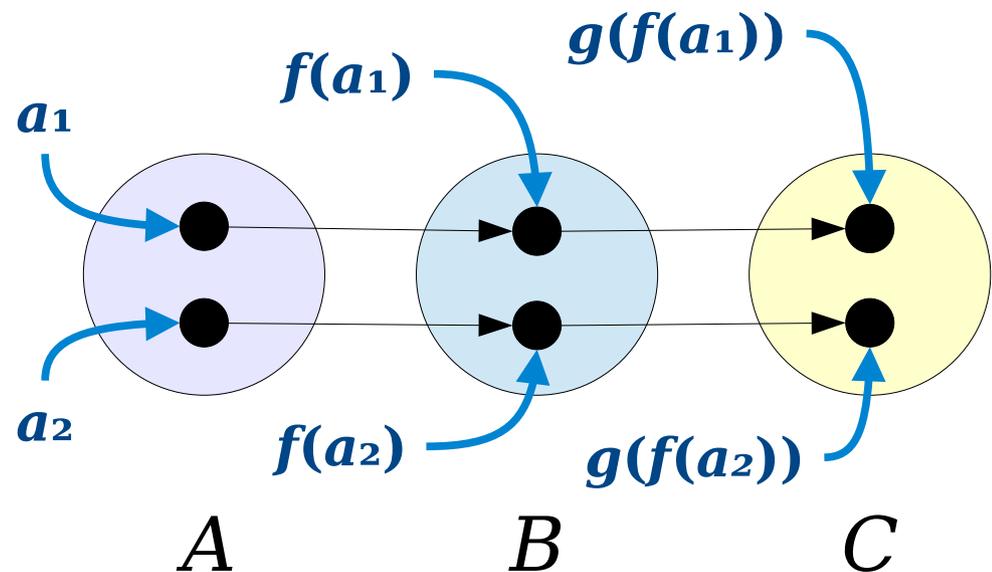


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Great exercise: Repeat this proof using the other definition of injectivity.

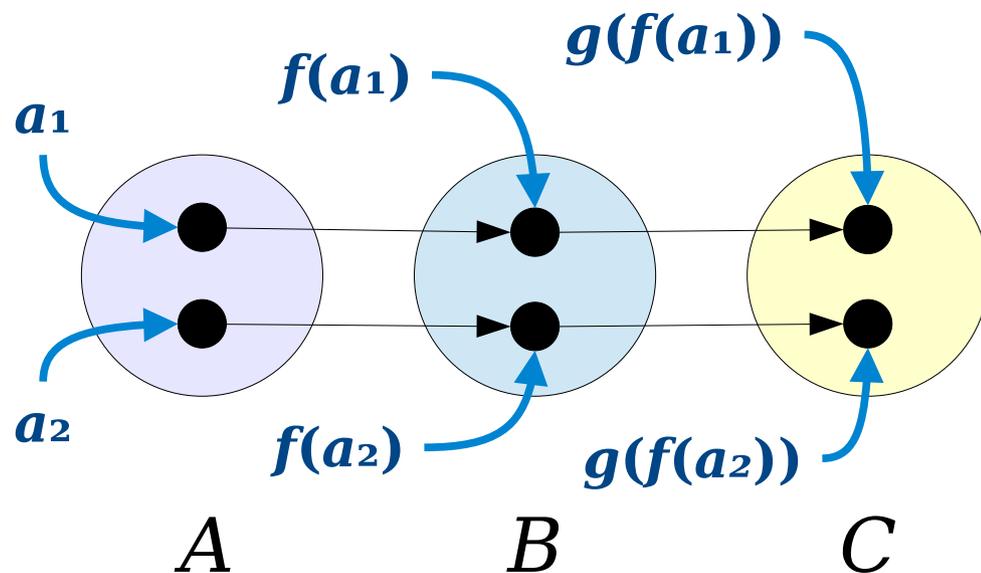


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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.



Theorem: If $f : A \rightarrow B$ is a surjection and $g : B \rightarrow C$ is a surjection, then the function $g \circ f : A \rightarrow C$ is a surjection.

Proof: In the appendix!

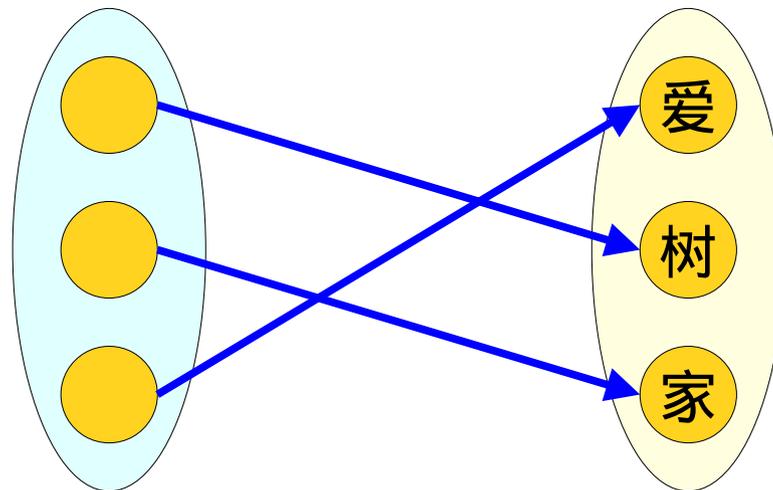
Bijections

Injections and Surjections

- An injective function associates *at most* one element of the domain with each element of the codomain.
- A surjective function associates *at least* one element of the domain with each element of the codomain.
- What about functions that associate *exactly one* element of the domain with each element of the codomain?

Bijections

- A ***bijection*** is a function that is both injective and surjective.
- Intuitively, if $f : A \rightarrow B$ is a bijection, then f represents a way of pairing off elements of A and elements of B .



Bijections

- Which of the following are bijections?
 - $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = x$.
 - $f : \mathbb{Z} \rightarrow \mathbb{R}$ defined as $f(x) = x$.
 - $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = 2x + 1$.
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Cardinality Revisited

Cardinality

- Recall (*from our first lecture!*) that the **cardinality** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by $|S|$.
- For finite sets, cardinalities are natural numbers:
 - $|\{1, 2, 3\}| = 3$
 - $|\{100, 200\}| = 2$
- For infinite sets, we introduced **infinite cardinals** to denote the size of sets:

$$|\mathbb{N}| = \aleph_0$$

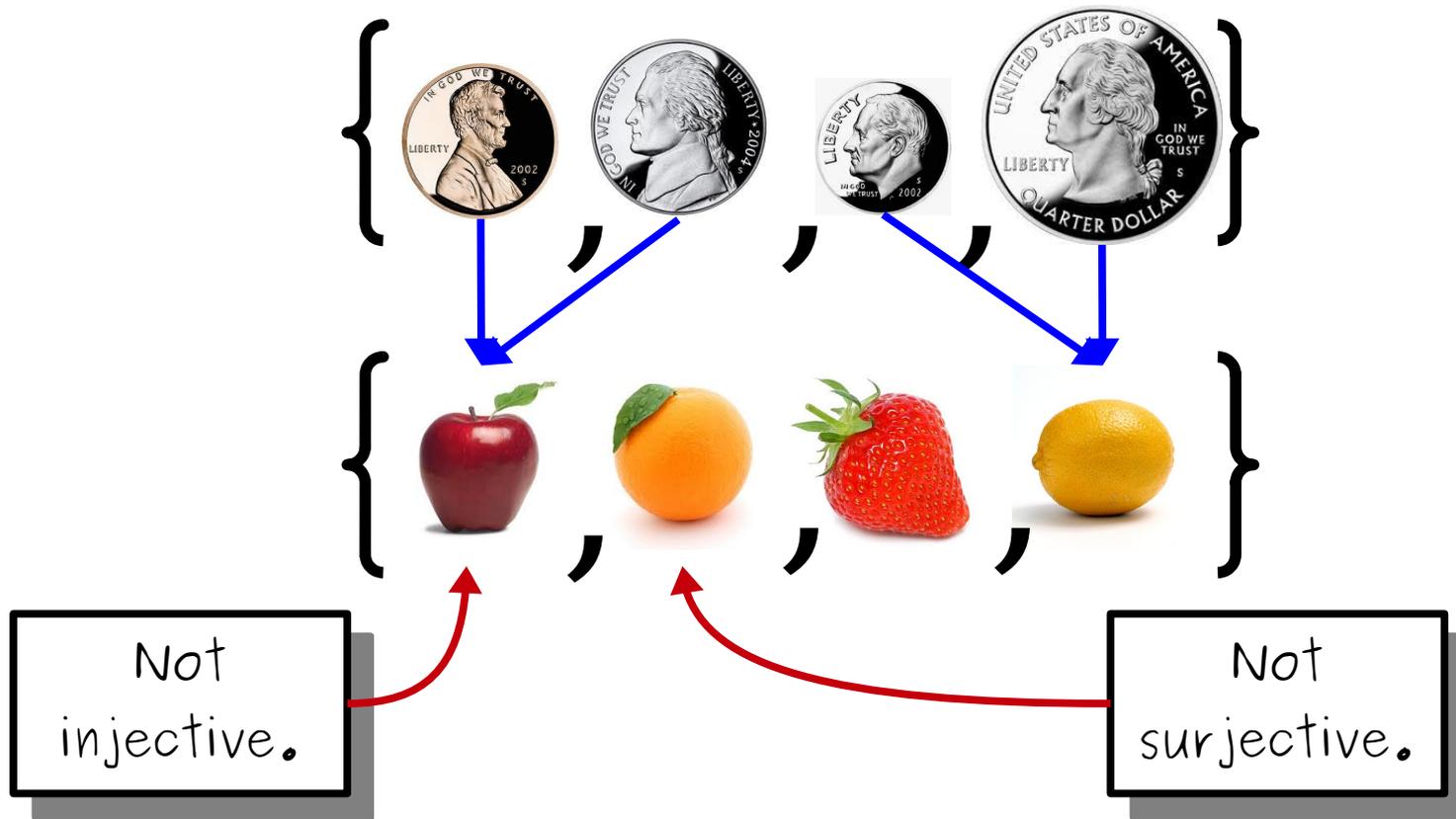
Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
 - What is 4? What is \aleph_0 ?
 - (Take Math 161 for an answer!)
- Instead, we'll define cardinality as a *relation* between two sets rather than an absolute quantity.
- ***Intuition:*** Two sets have the same cardinality if there's a way to pair off their elements.

Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

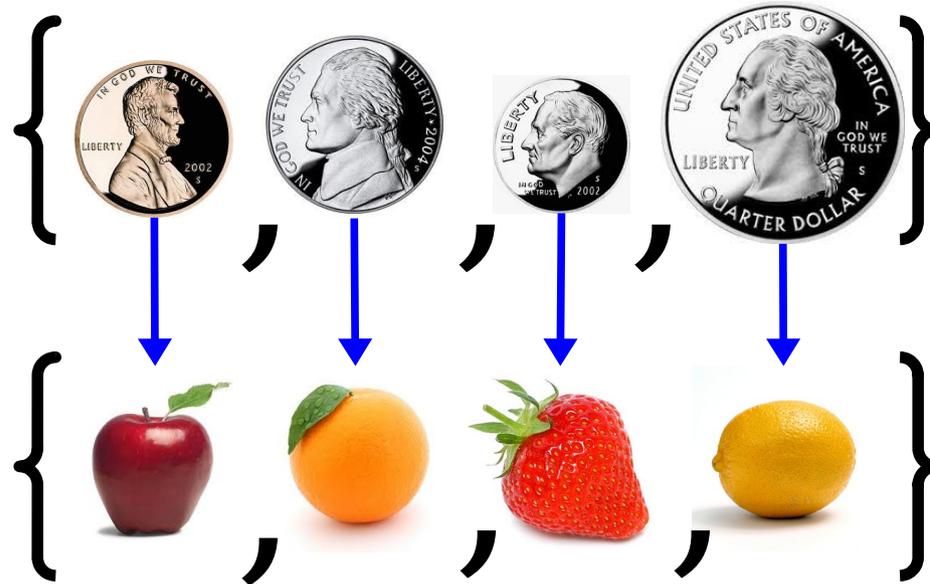
$|S| = |T|$ if there exists a *bijection* $f : S \rightarrow T$



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Fun with Cardinality

Terminology Refresher

- Let a and b be real numbers where $a \leq b$.
- The notation $[a, b]$ denotes the set of all real numbers between a and b , inclusive.

$$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

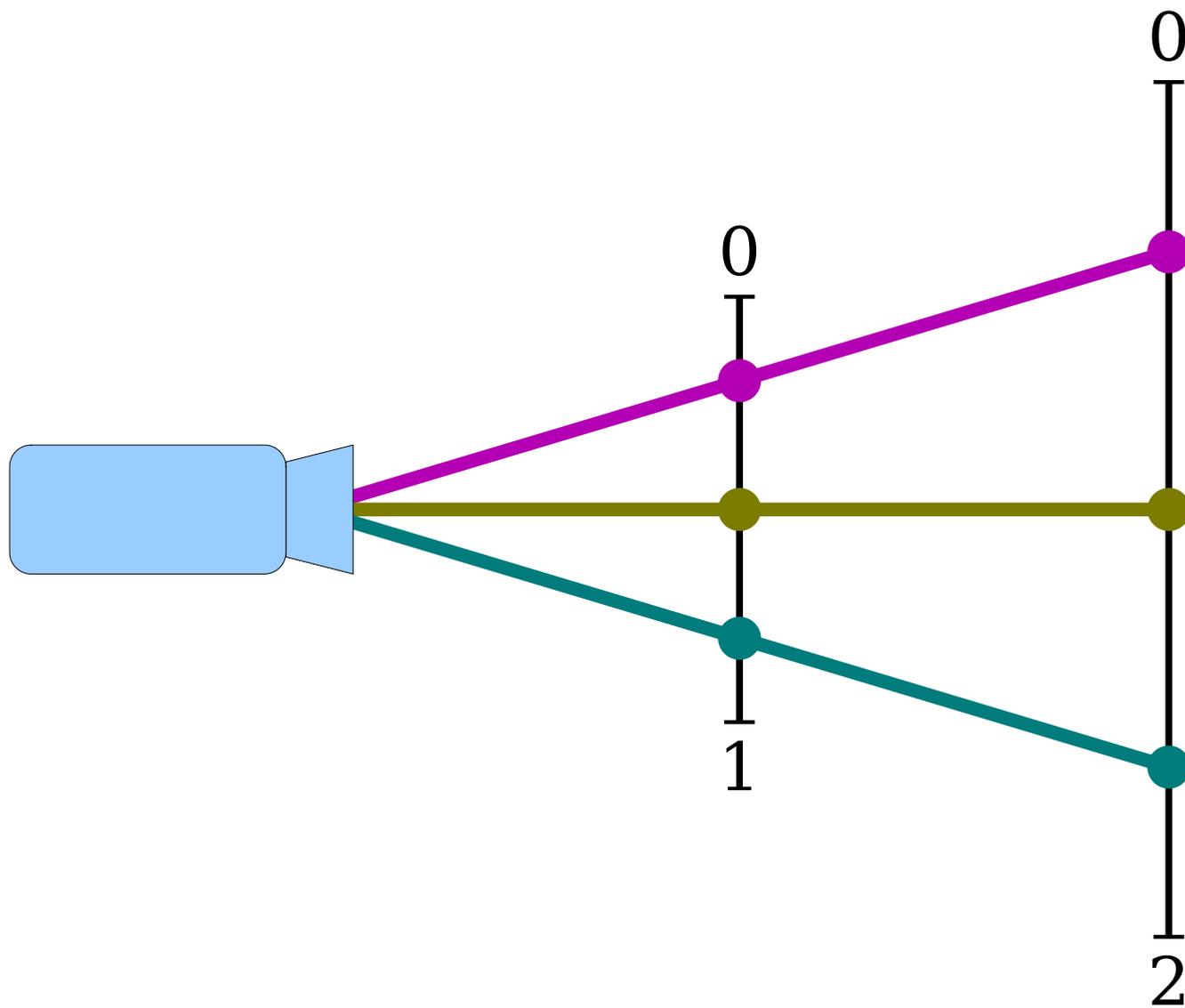
- The notation (a, b) denotes the set of all real numbers between a and b , exclusive.

$$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$$

Consider the sets $[0, 1]$ and $[0, 2]$.

How do their cardinalities compare?





$f : [0, 1] \rightarrow [0, 2]$
 $f(x) = 2x$

Theorem: $|[0, 1]| = |[0, 2]|$.

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$$f(x) = 2x = 2(y/2) = y,$$

so $f(x) = y$, as required.

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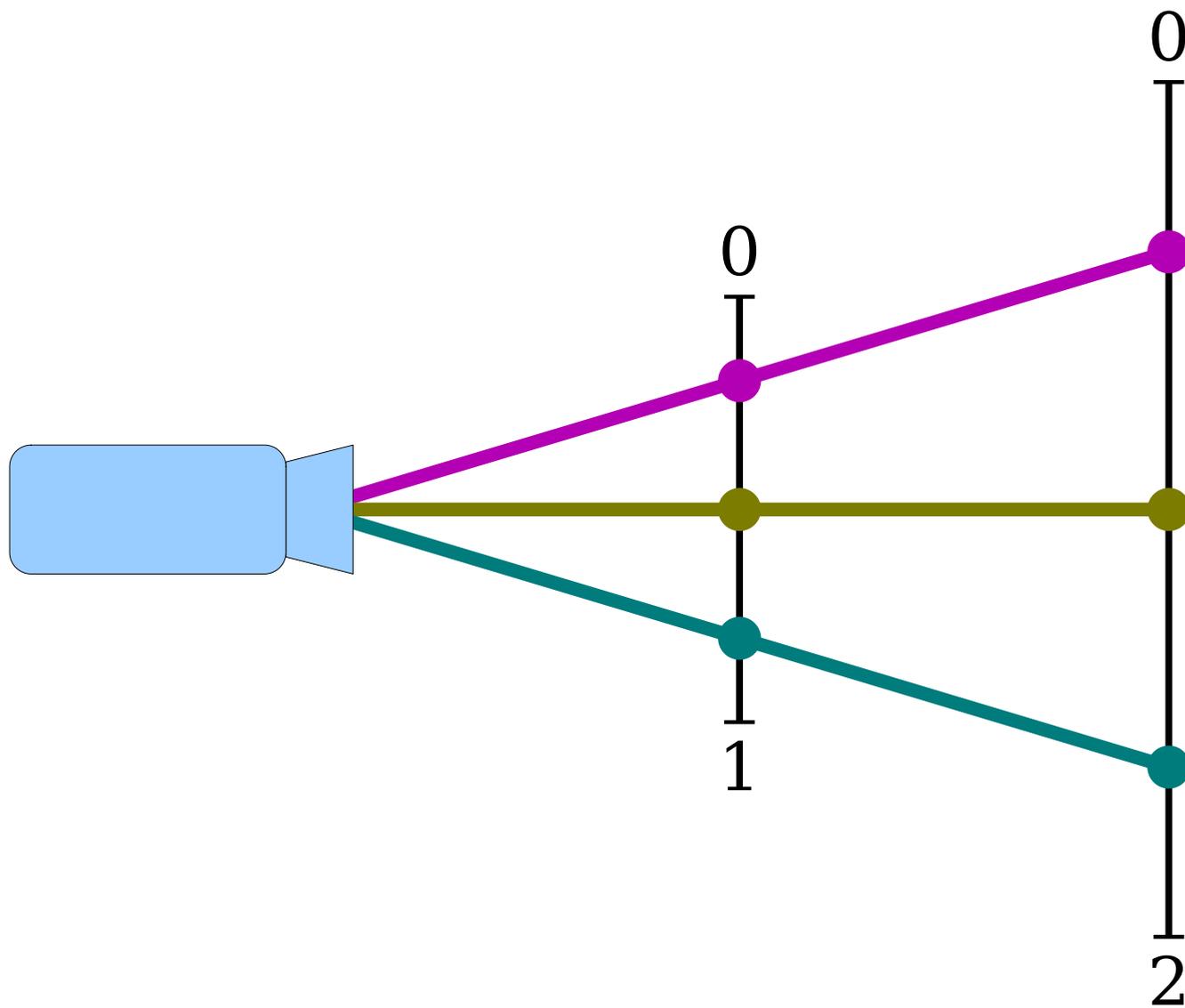
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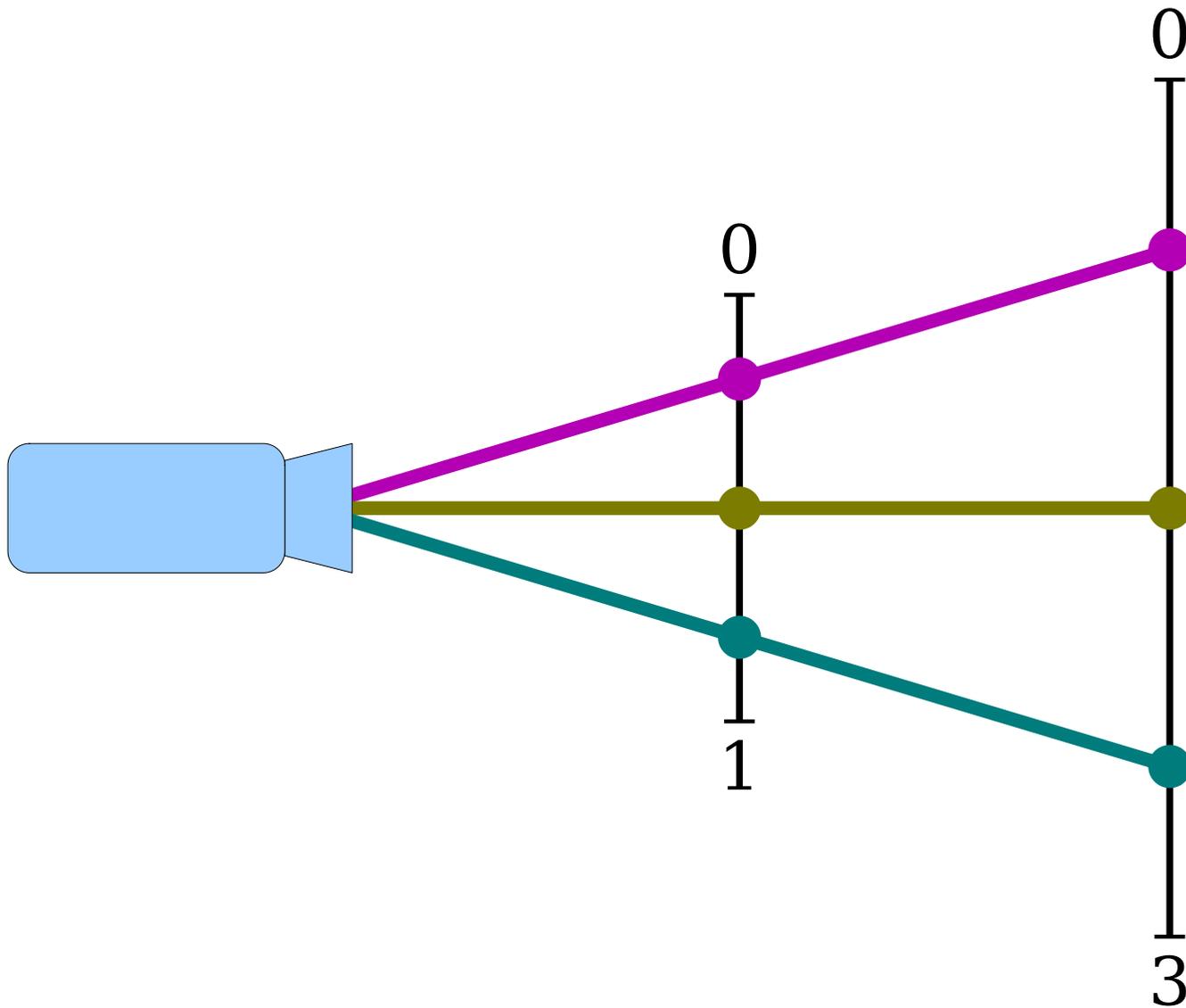
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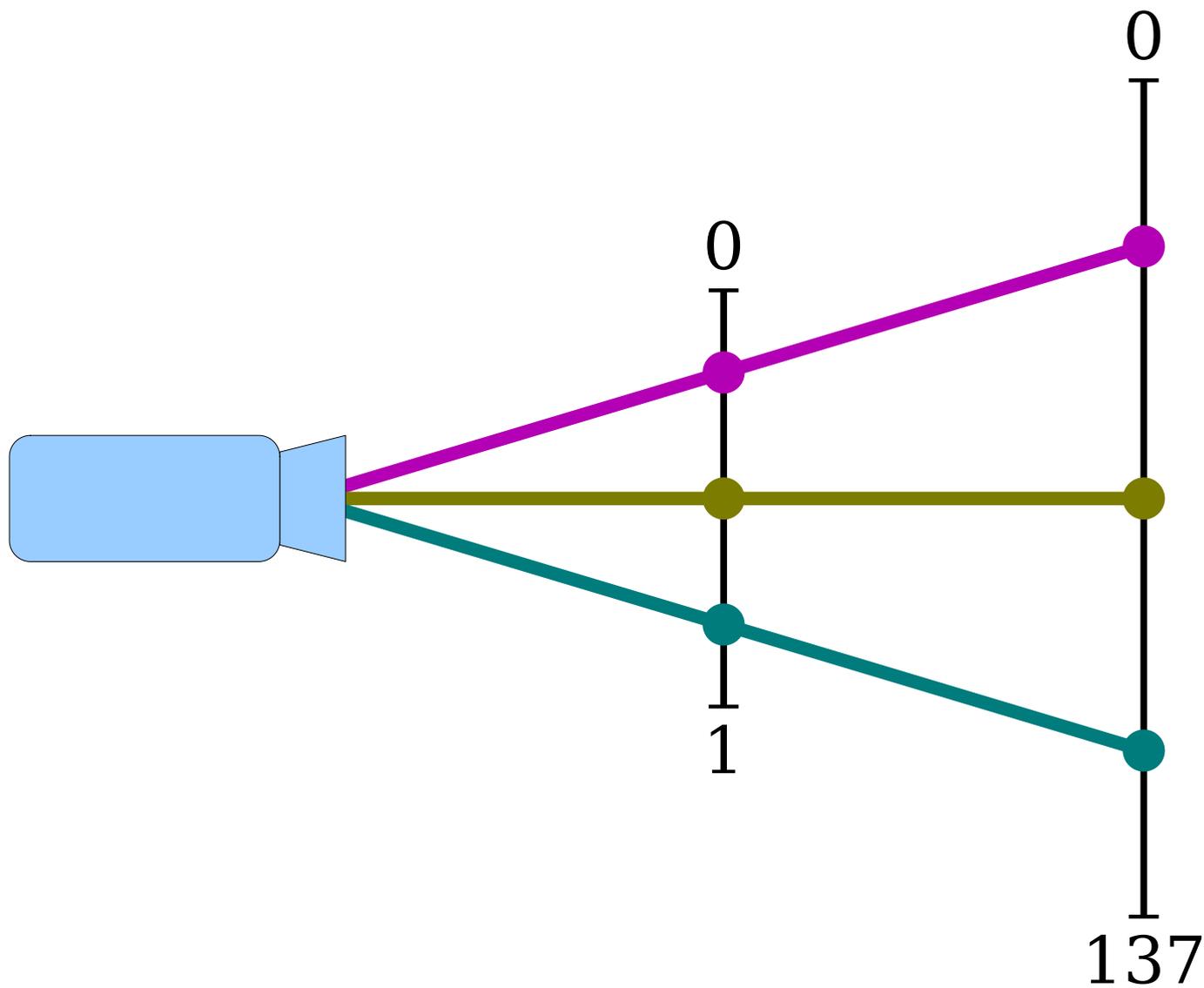
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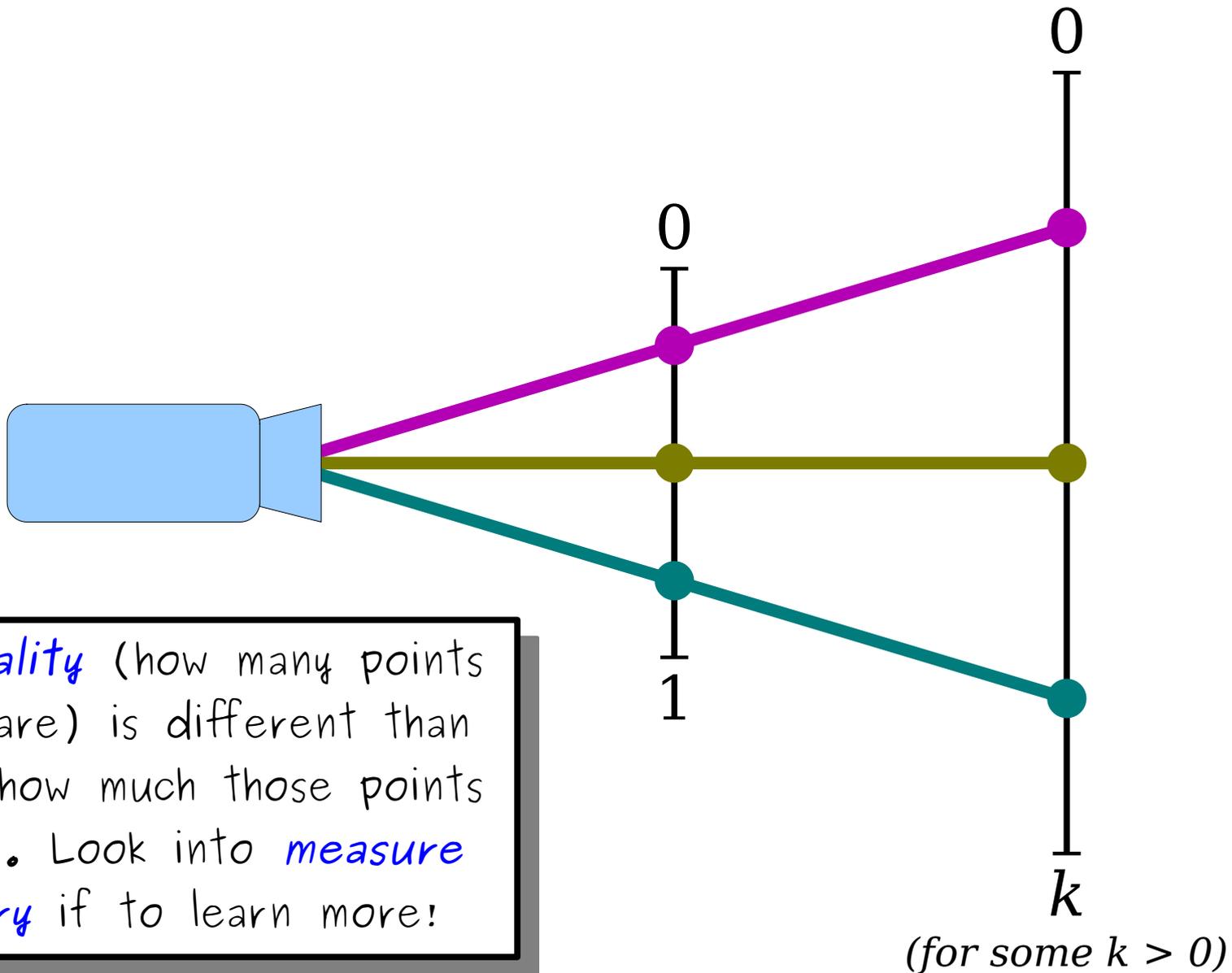
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$$f : [0, 1] \rightarrow [0, 3]$$
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$$f : [0, 1] \rightarrow [0, 137]$$
$$f(x) = 137x$$

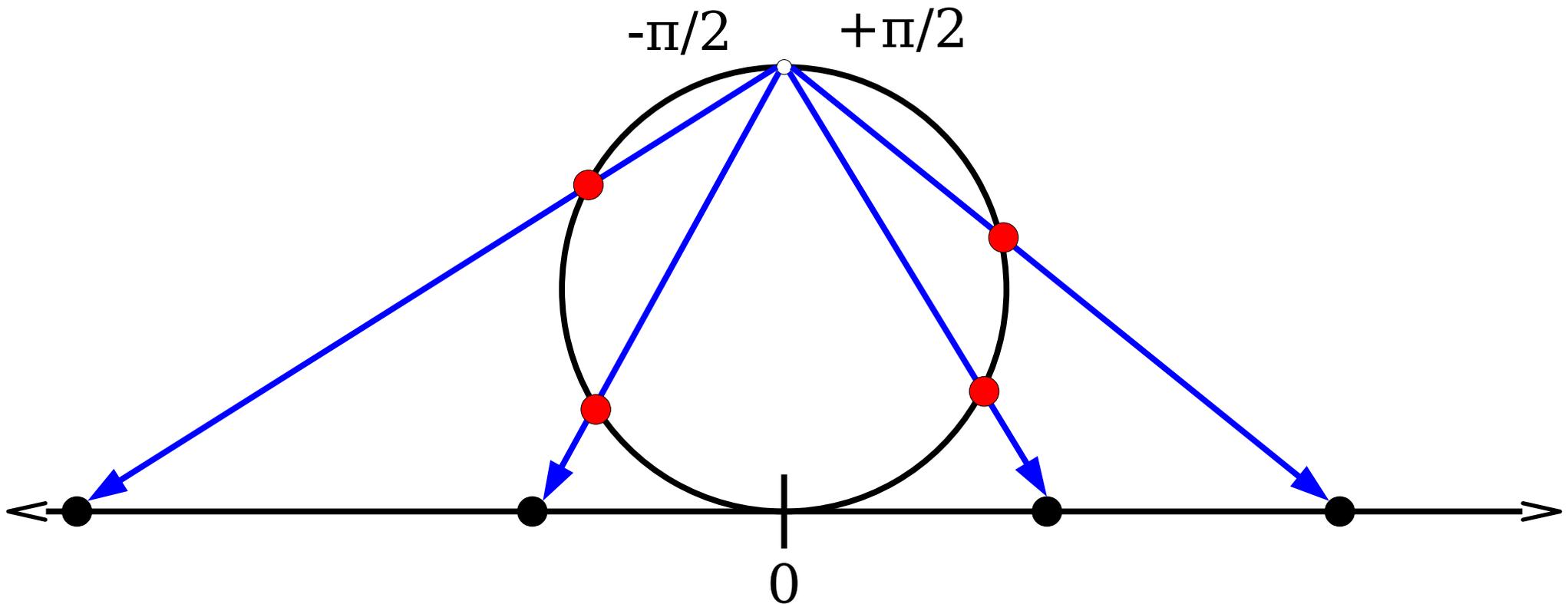


Cardinality (how many points there are) is different than *mass* (how much those points weigh). Look into *measure theory* if to learn more!

$$f : [0, 1] \rightarrow [0, k]$$
$$f(x) = kx$$

And one more example, just for funzies.

Put a Ring On It



$$f : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$$
$$f(x) = \tan x$$

$$|(-\pi/2, \pi/2)| = |\mathbb{R}|$$

Facts About Cardinality

- **Theorem:** For any set A , we have $|A| = |A|$.
 - **Proof Idea:** Define a bijection from A to itself. Specifically, pick $f(x) = x$.
 - **Proof:** Appendix!
- **Theorem:** For any sets A , B , and C , if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.
 - **Proof Idea:** We begin with bijections $f : A \rightarrow B$ and $g : B \rightarrow C$. Compose them to get a bijection $g \circ f : A \rightarrow C$.
 - **Proof:** Appendix!
- **Theorem:** For any sets A and B , if $|A| = |B|$, then $|B| = |A|$.
 - **Proof Idea:** Start with a bijection $f : A \rightarrow B$ and look at its **inverse**, the function $f^{-1} : B \rightarrow A$, which is also a bijection.
 - **Proof:** Appendix!

Time-Out for Announcements!

Problem Set Three

- Problem Set Two was due today at 2:30PM.
- Problem Set Three goes out today. It's due next Friday at 2:30PM.
 - Play around with functions, set cardinality, and the nature of infinity!
- As always, ping us if you need help working on this one: post on EdStem or stop by office hours.

Extra Practice Problems

- We've posted a bank of 24 cumulative practice problems to the course website, along with solutions.
- Feel free to use these as an extra study resource to get more reps with the material.
- You can also use this to study for the midterm if you want to get a jump on that. More details on Monday.

Alternate Exams

- We are working on finalizing alternate exam times for the first midterm exam.
- If you have an OAE letter and haven't yet contacted us, please do so ASAP so we can reserve space for you.
- If you have an academic conflict during the normal exam time, please let us know as soon as possible.

Your Questions

“What is your affinity for the number 137?”

It's a fun example of a "nothing up my sleeve number" that has a fun historical backstory.

Back to CS103!

Unequal Cardinalities

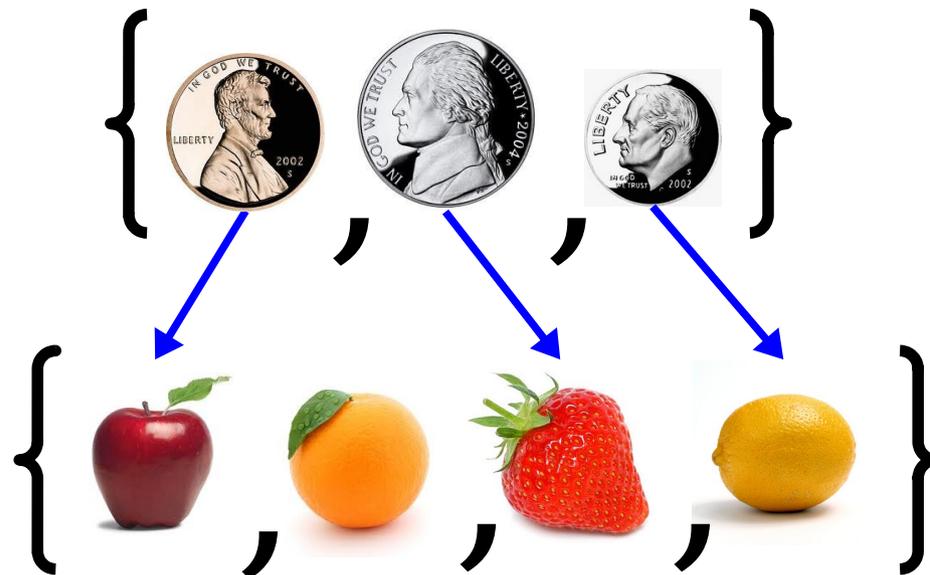
- Recall: $|A| = |B|$ if the following statement is true:

There exists a bijection $f : A \rightarrow B$

- What does it mean for $|A| \neq |B|$ to be true?

Every function $f : A \rightarrow B$ is not a bijection.

- This is a strong statement! To prove $|A| \neq |B|$, we need to show that *no possible function* from A to B can be injective and surjective.



Unequal Cardinalities

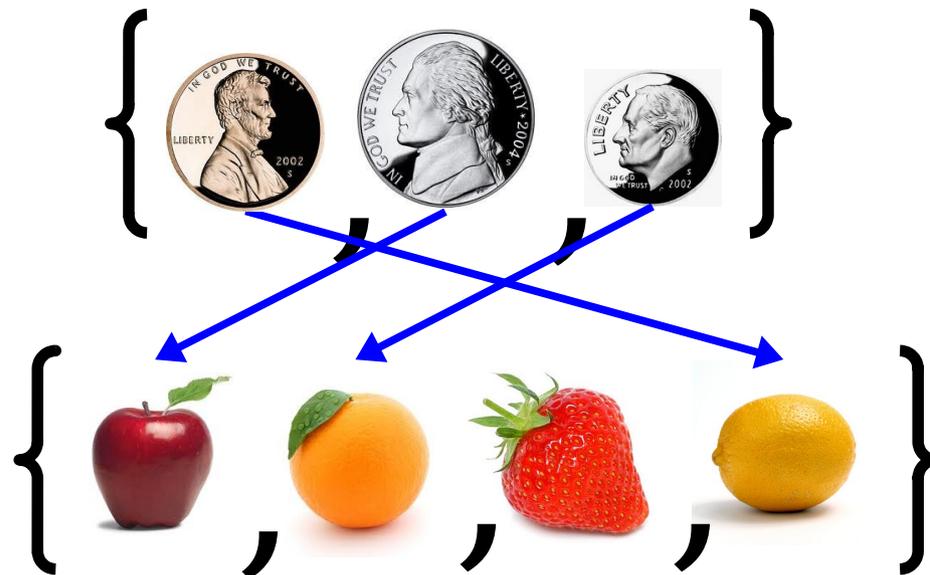
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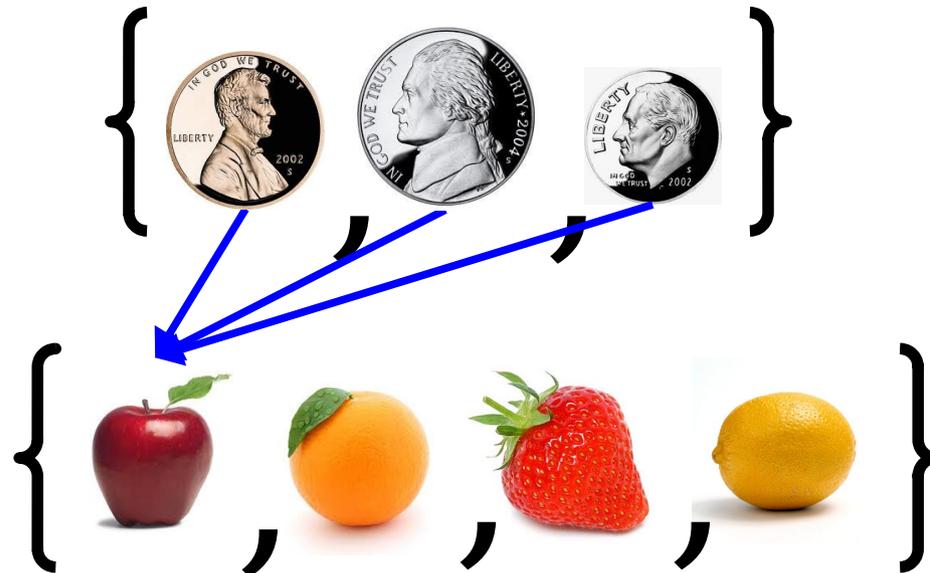
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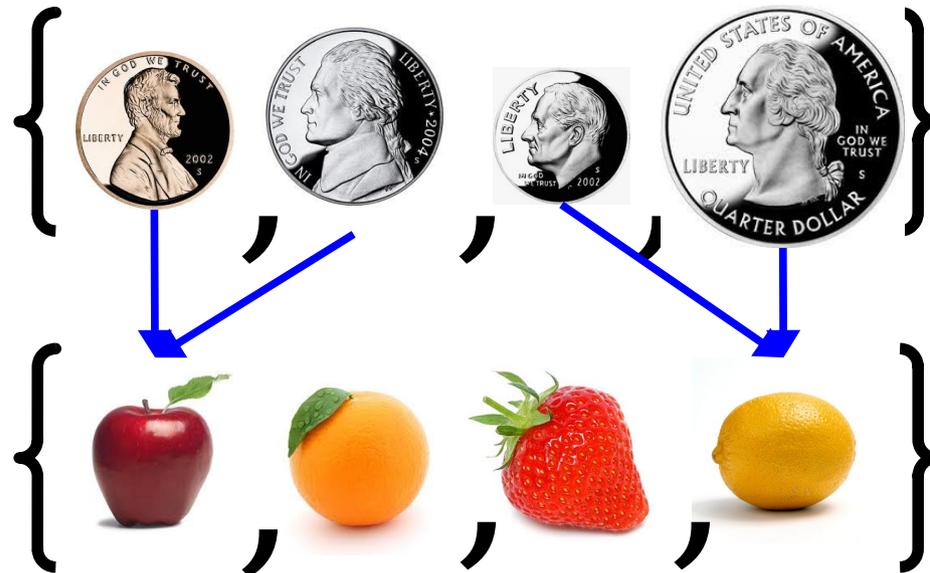
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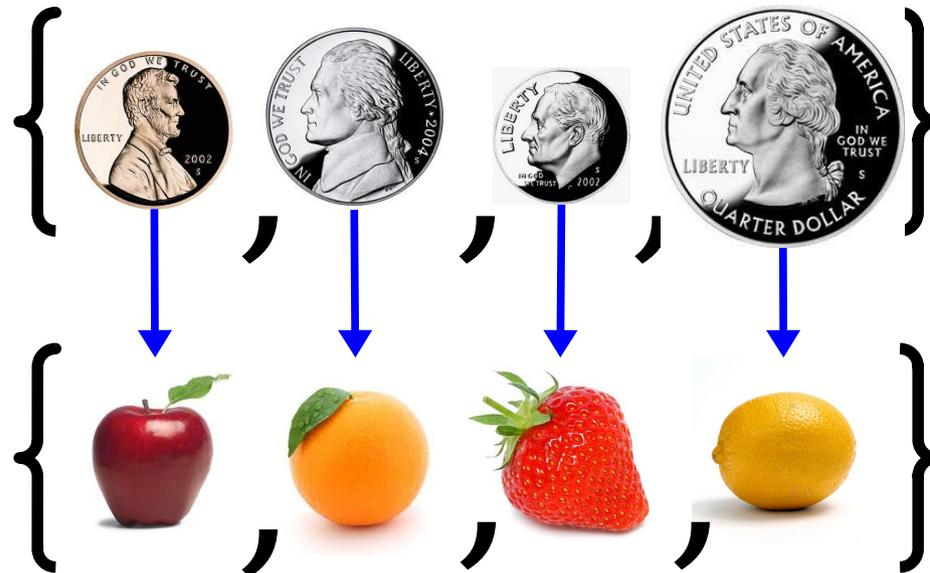
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Cantor's Theorem Revisited

Cantor's Theorem

- In our very first lecture, we sketched out a proof of ***Cantor's theorem***, which says that

If S is a set, then $|S| < |\wp(S)|$.

- That proof was visual and pretty hand-wavy. Let's see if we can go back and formalize it!

Where We're Going

- Today, we're going to formally prove the following result:

If S is a set, then $|S| \neq |\mathcal{P}(S)|$.

- We've released an online Guide to Cantor's Theorem, which will go into *way* more depth than what we're going to see here.
- The goal for today will be to see how to start with our picture and turn it into something rigorous.
- On the problem set, you'll explore the proof in more depth and see some other applications.

The Roadmap

- We're going to prove this statement:

If S is a set, then $|S| \neq |\wp(S)|$.

- Here's how this will work:
 - Pick an arbitrary set S .
 - Pick an arbitrary function $f : S \rightarrow \wp(S)$.
 - Show that f is not surjective using a diagonal argument.
 - Conclude that there are no bijections from S to $\wp(S)$.
 - Conclude that $|S| \neq |\wp(S)|$.

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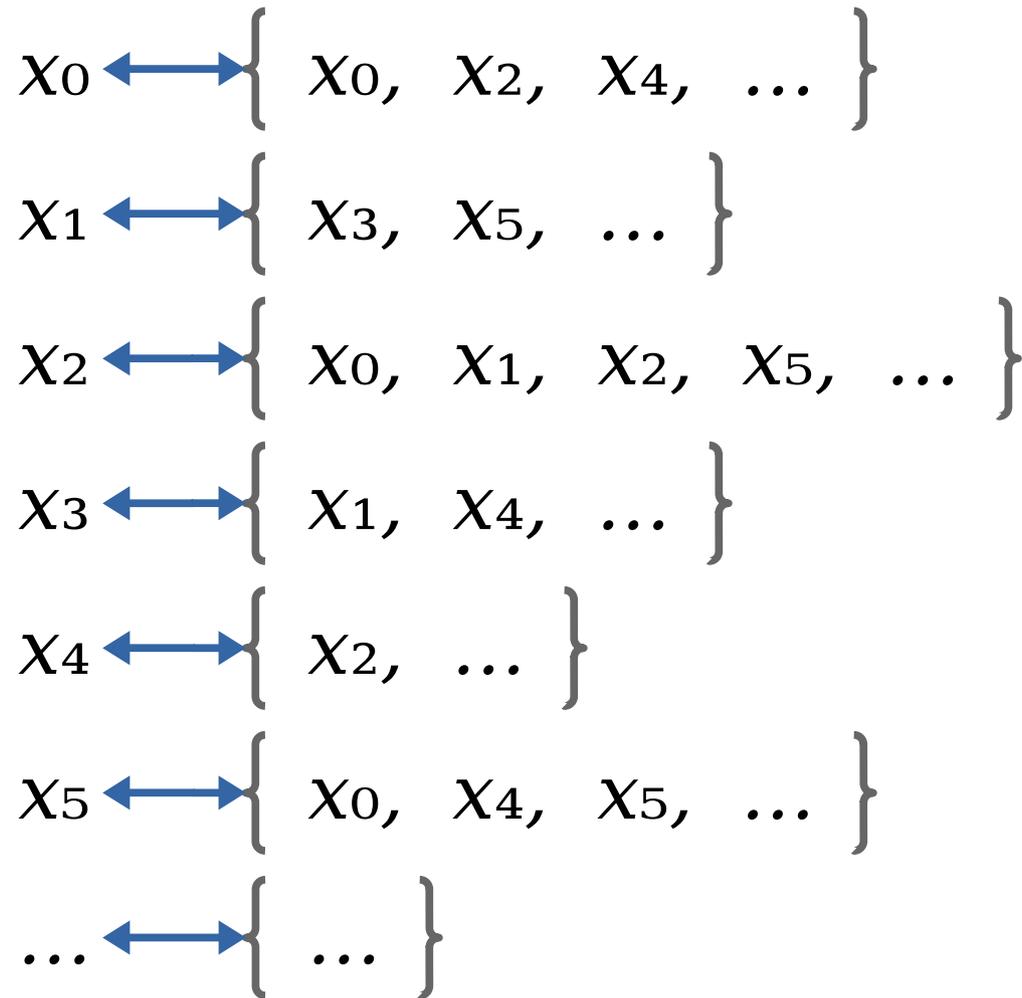
Pick an arbitrary function $f : S \rightarrow \wp(S)$.

- **Show that f is not surjective using a diagonal argument.**

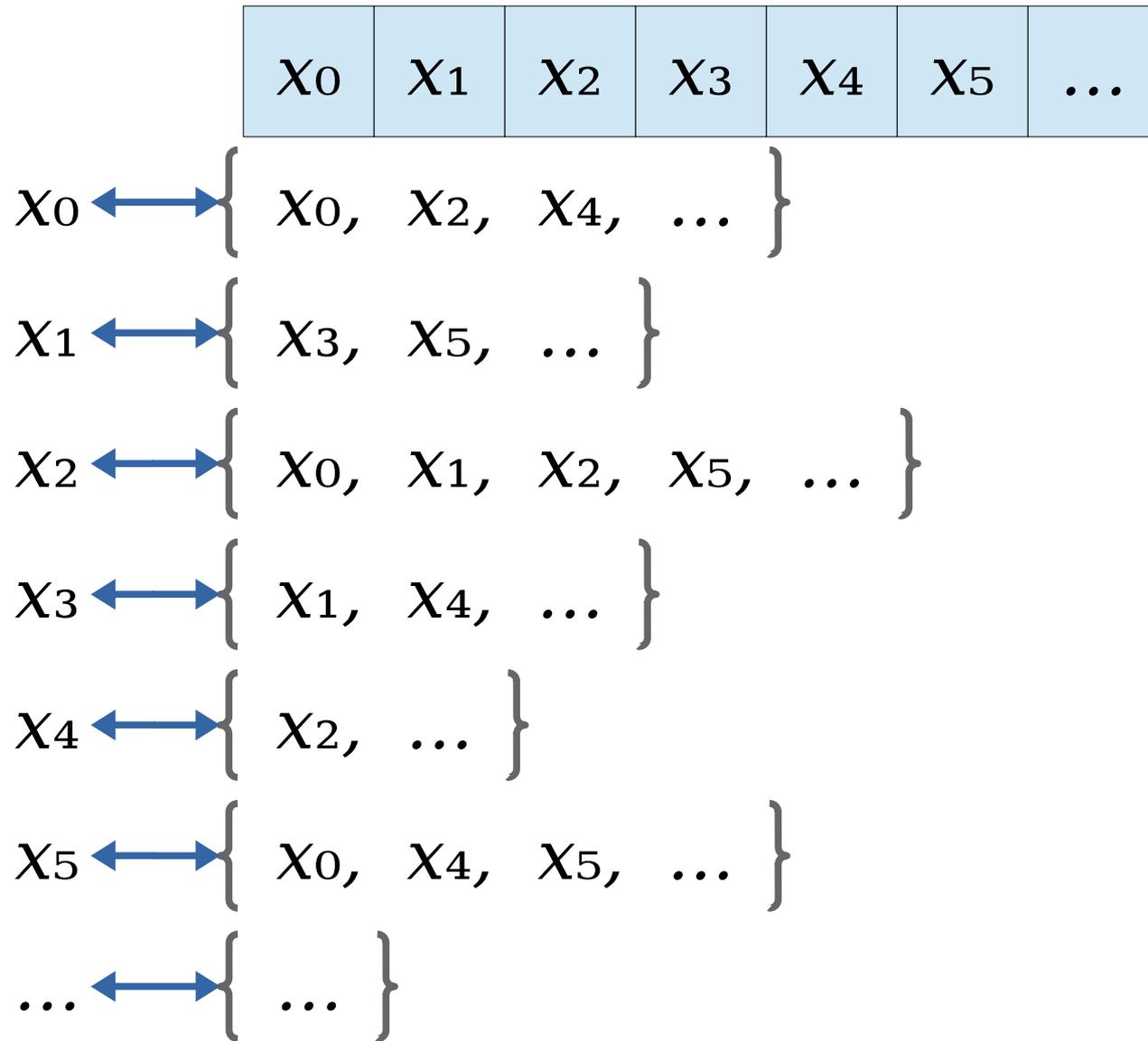
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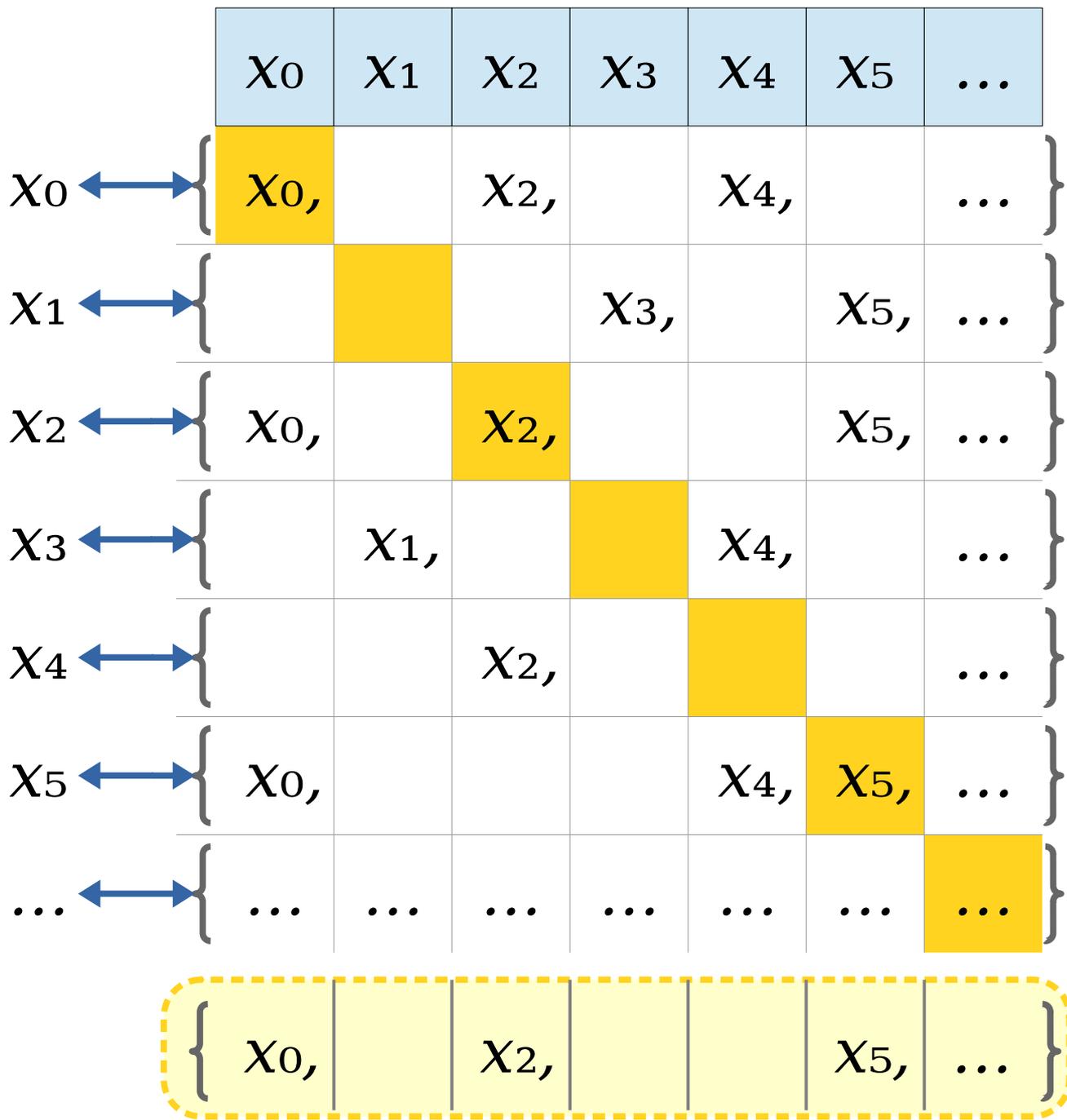
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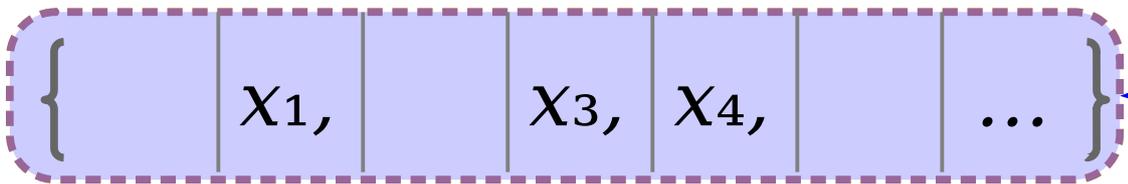
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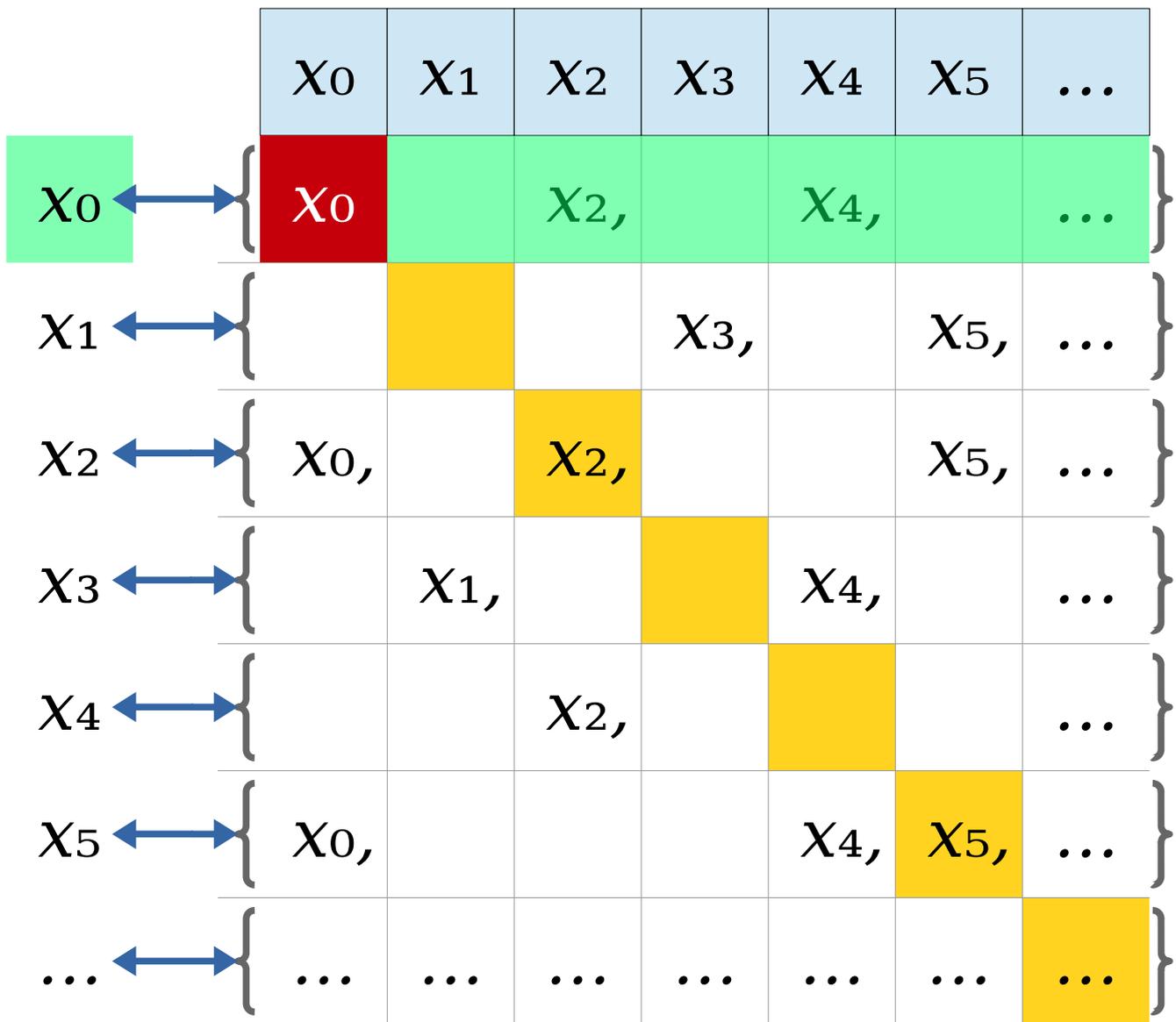
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	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	$x_0,$		$x_2,$		$x_4,$...
x_1				$x_3,$		$x_5,$...
x_2	$x_0,$		$x_2,$			$x_5,$...
x_3		$x_1,$			$x_4,$...
x_4			$x_2,$...
x_5	$x_0,$				$x_4,$	$x_5,$...
...

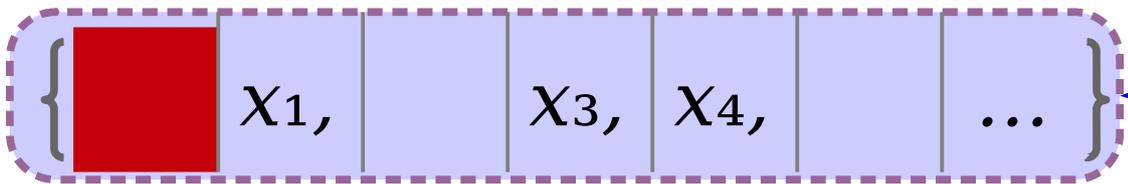
"Flip" this set. Swap what's included and what's excluded.



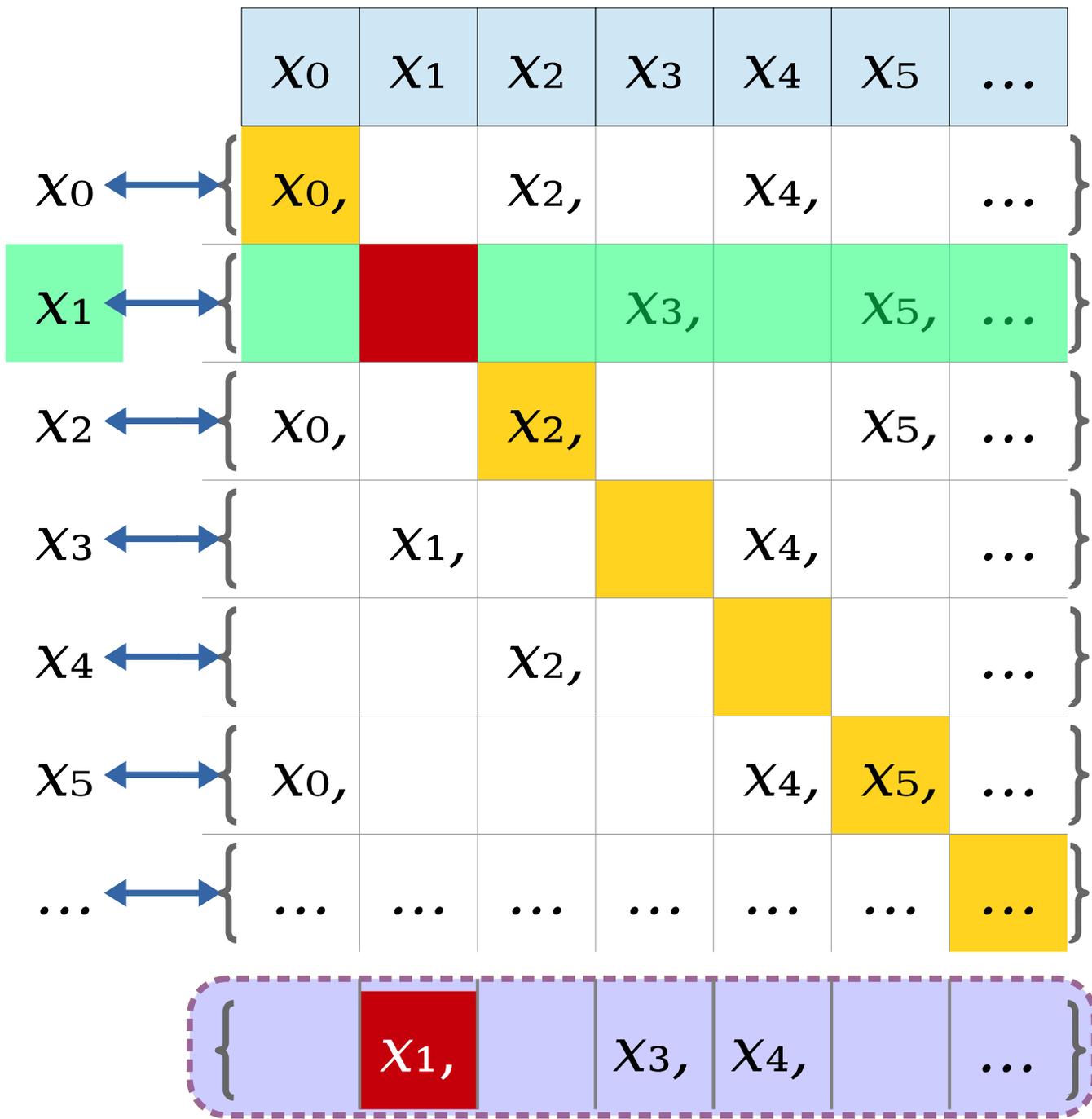
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Which element is paired with this set?

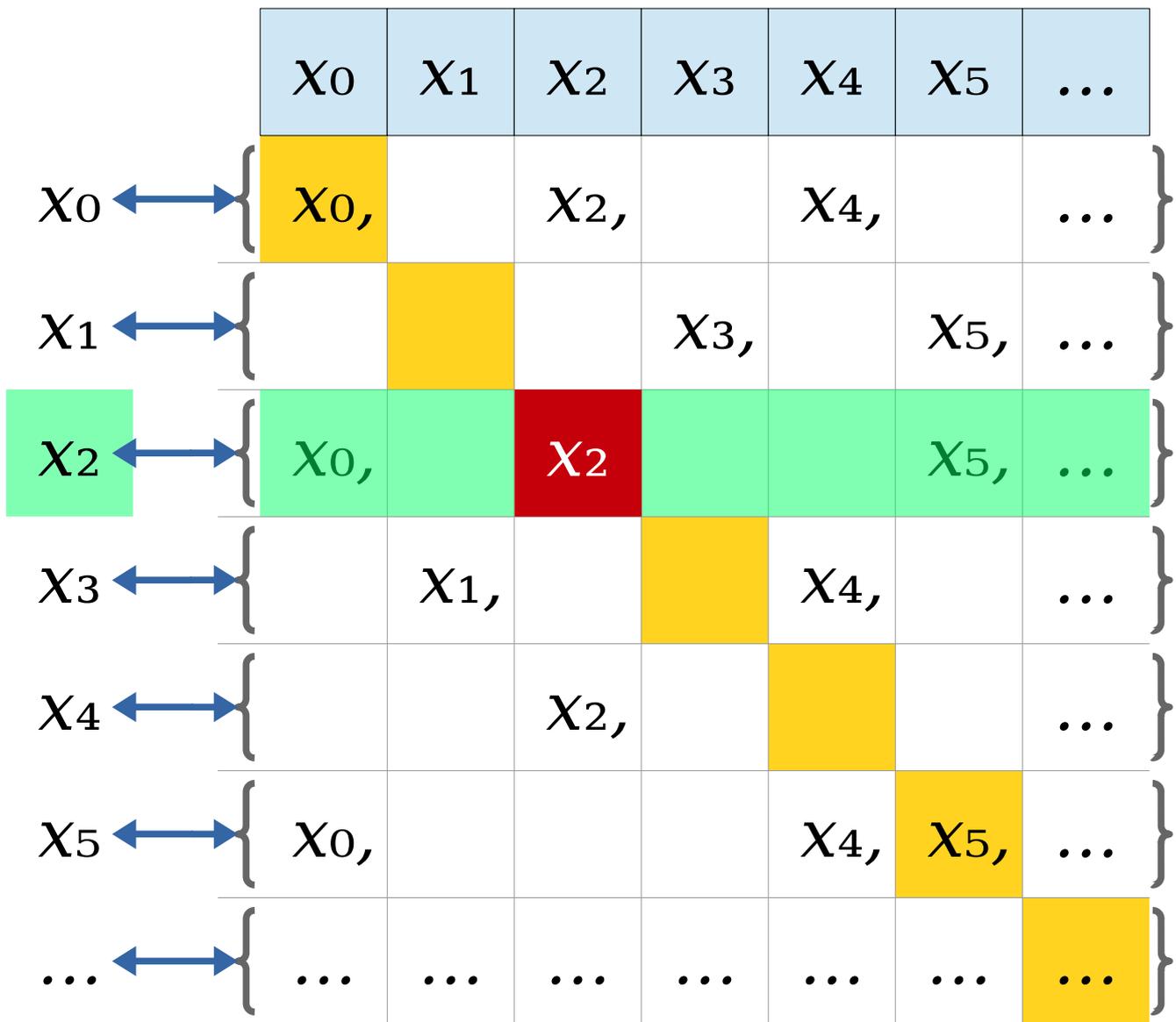


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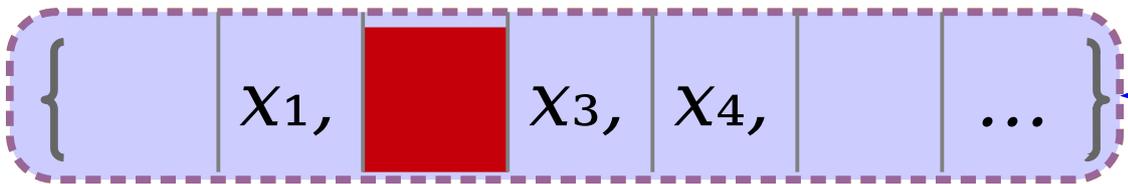


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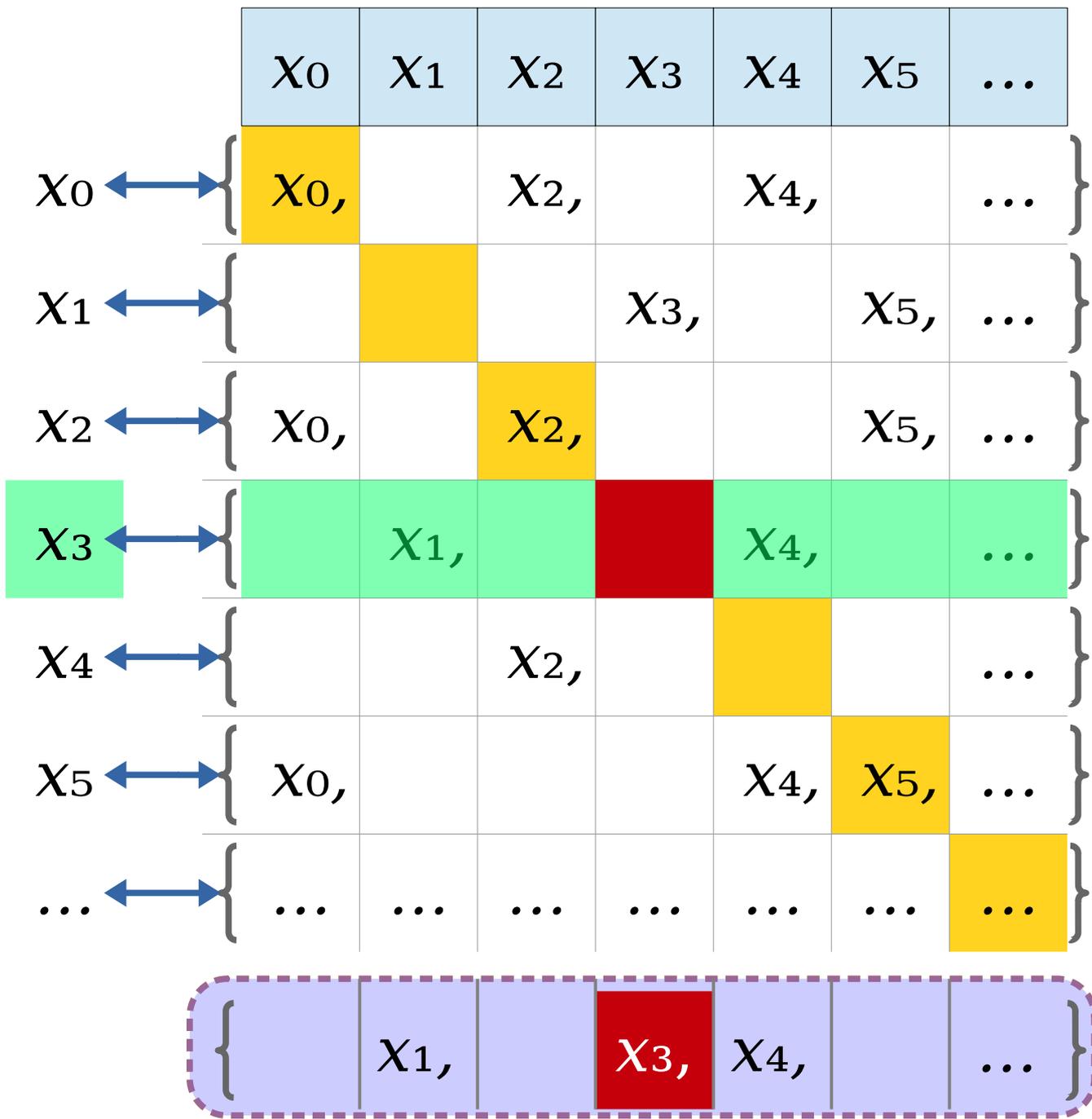
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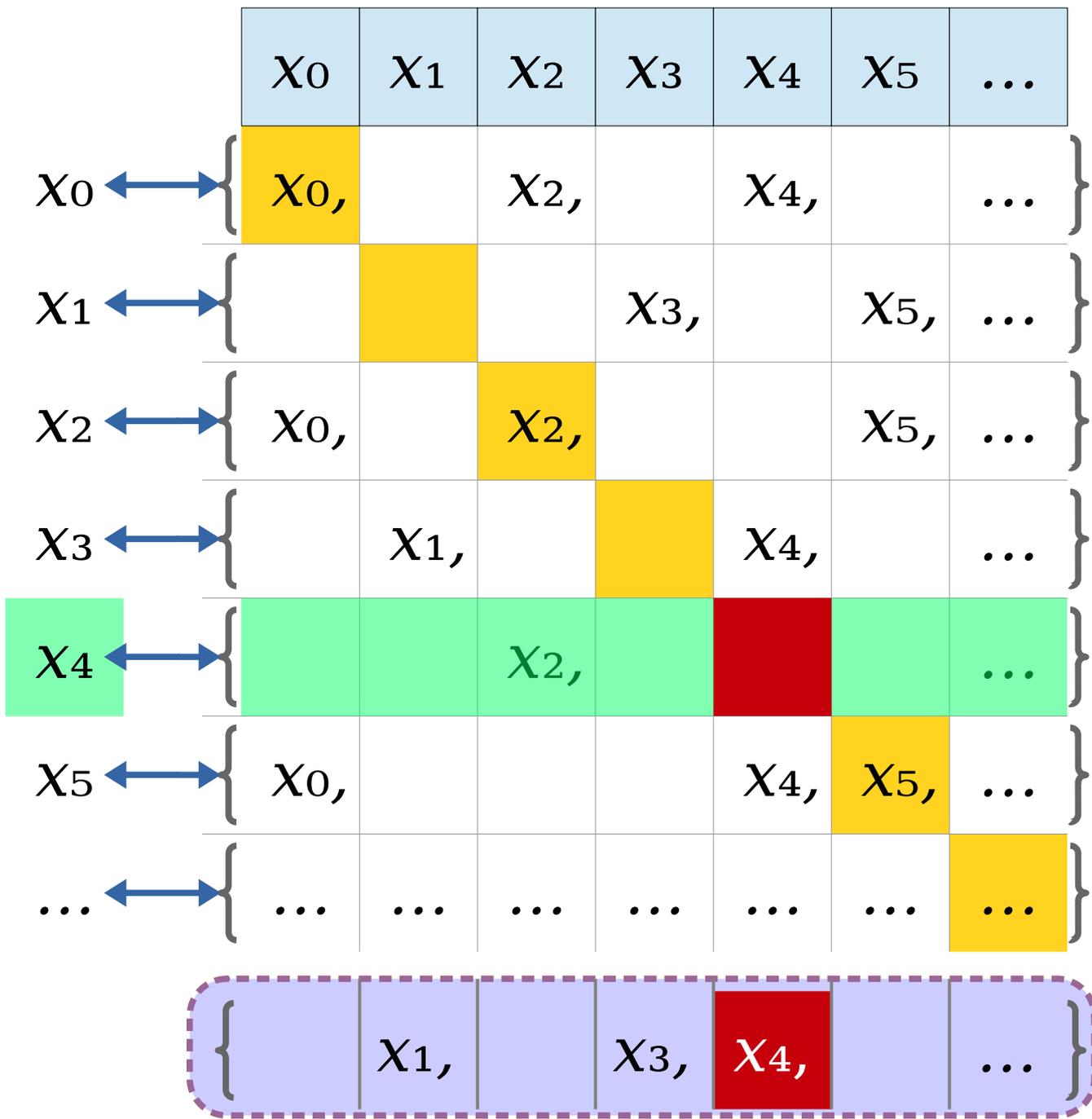


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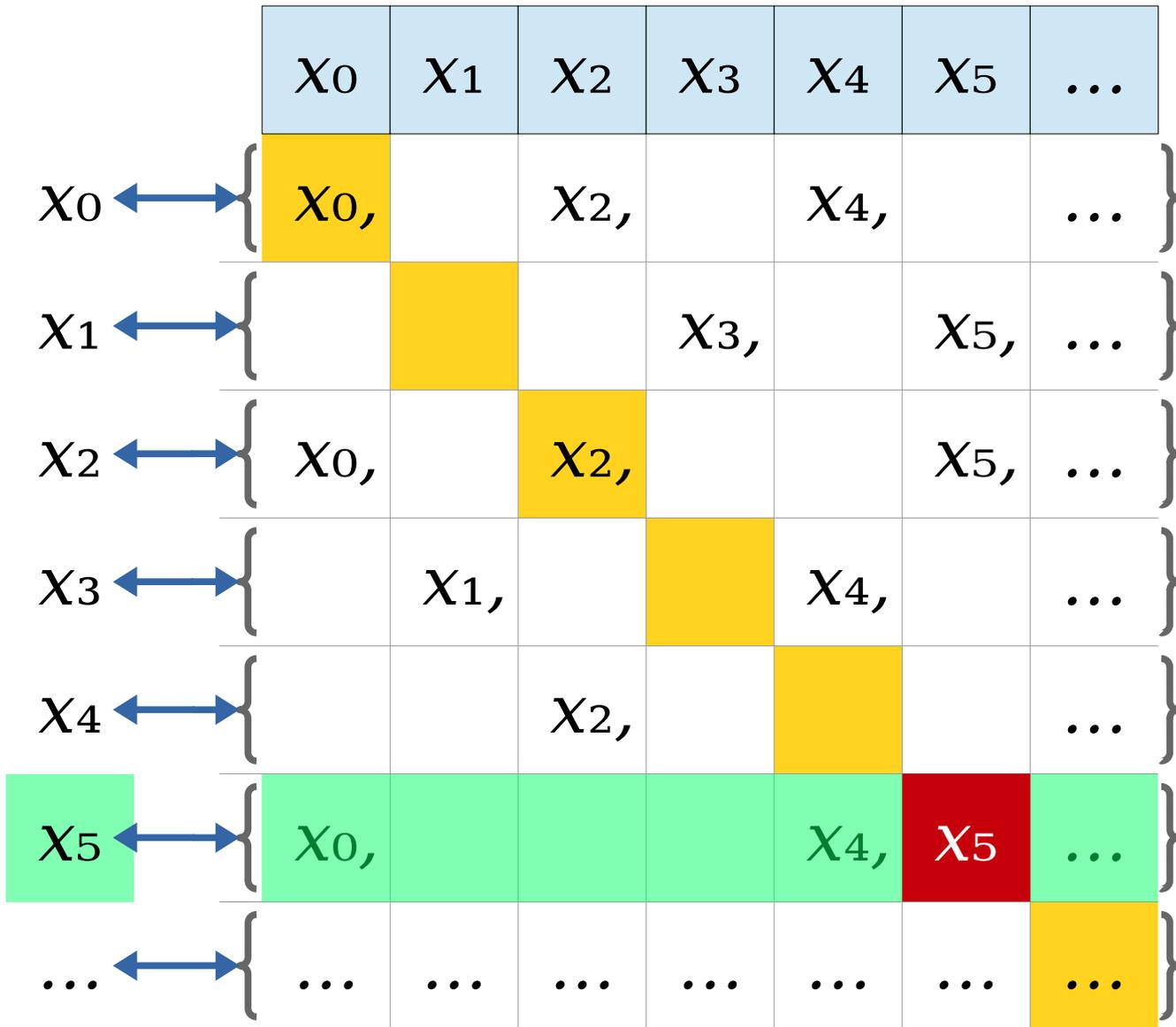
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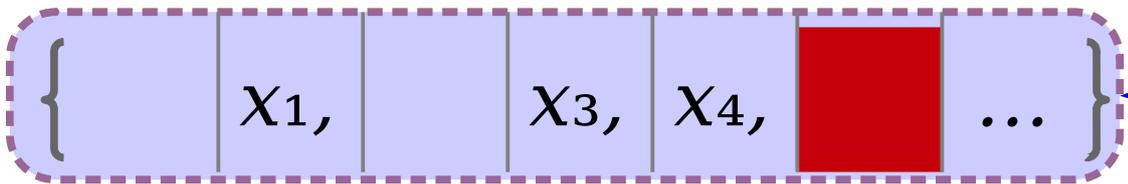


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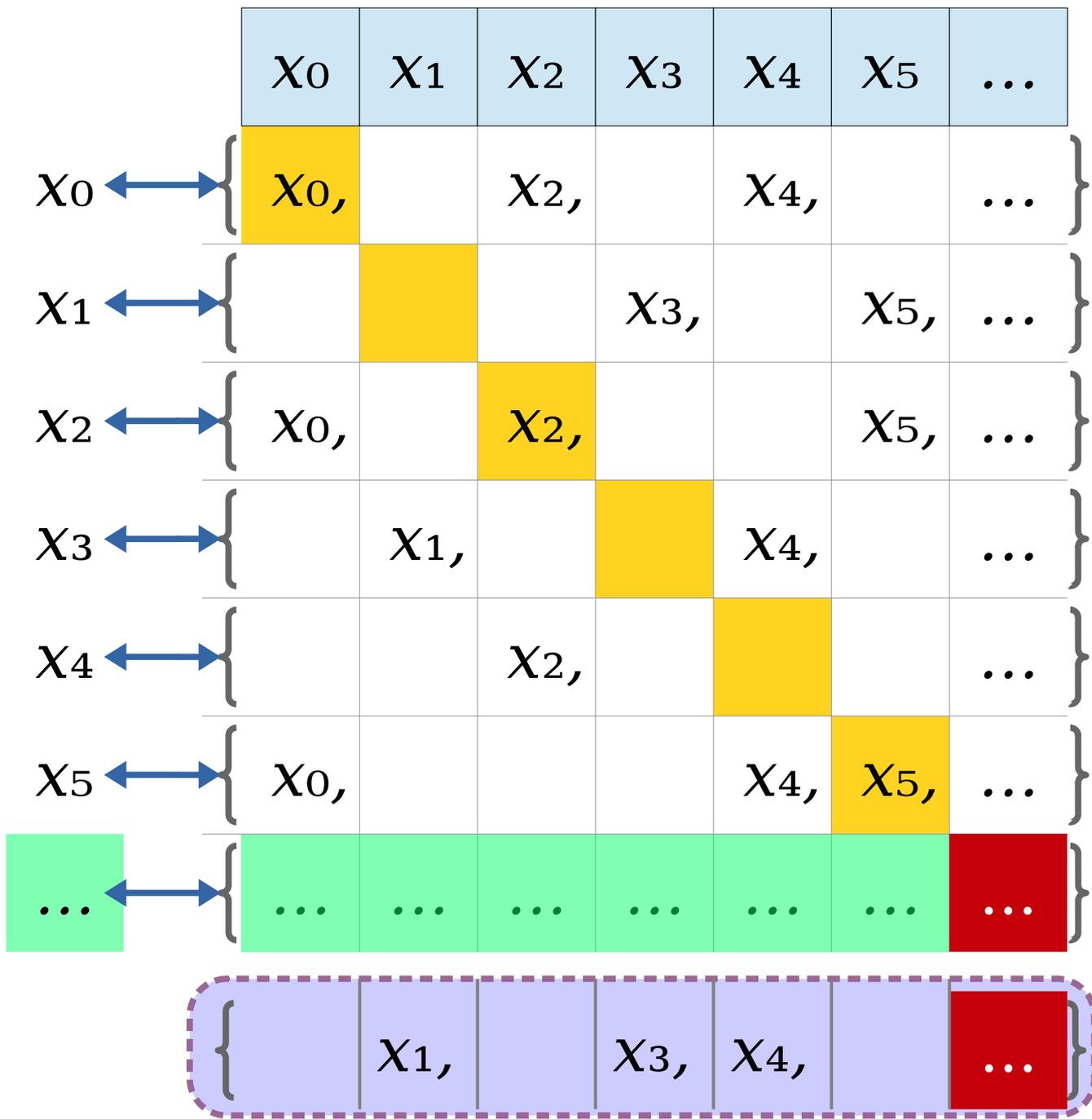
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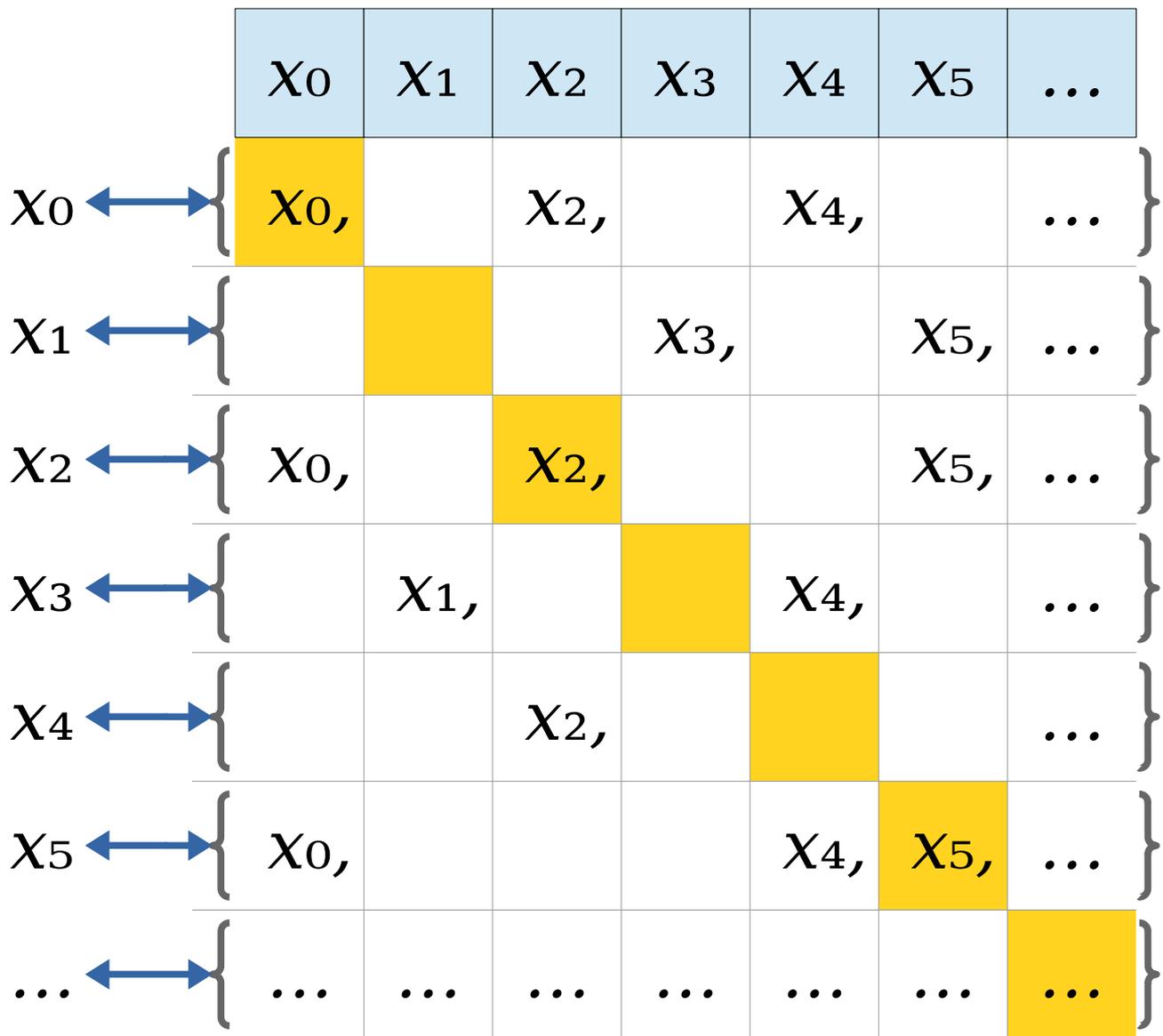


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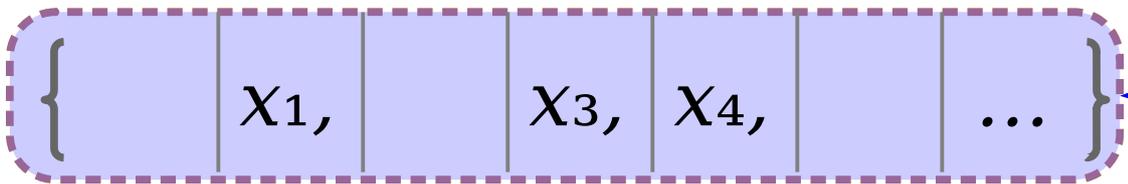


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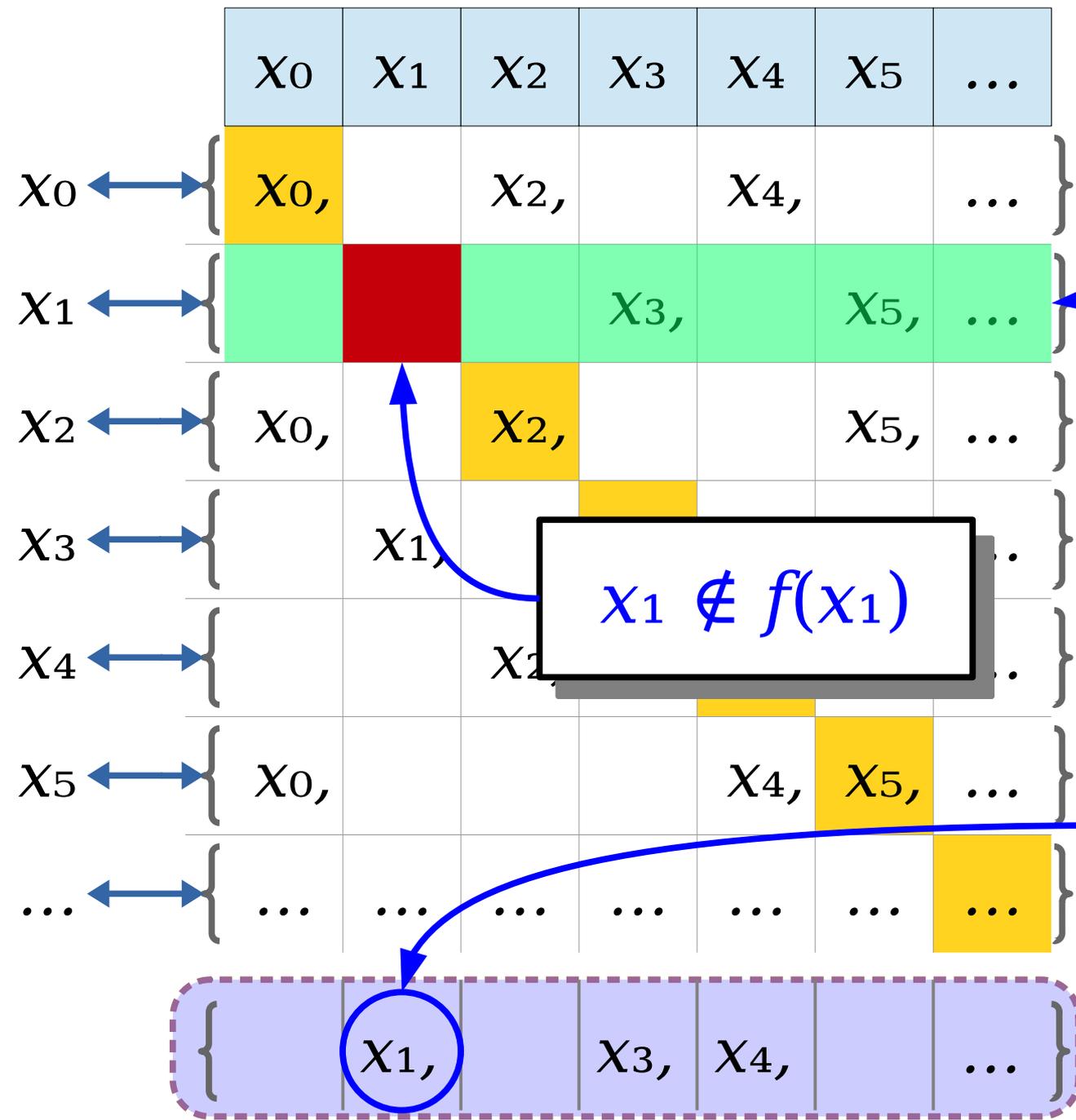


What set is this?



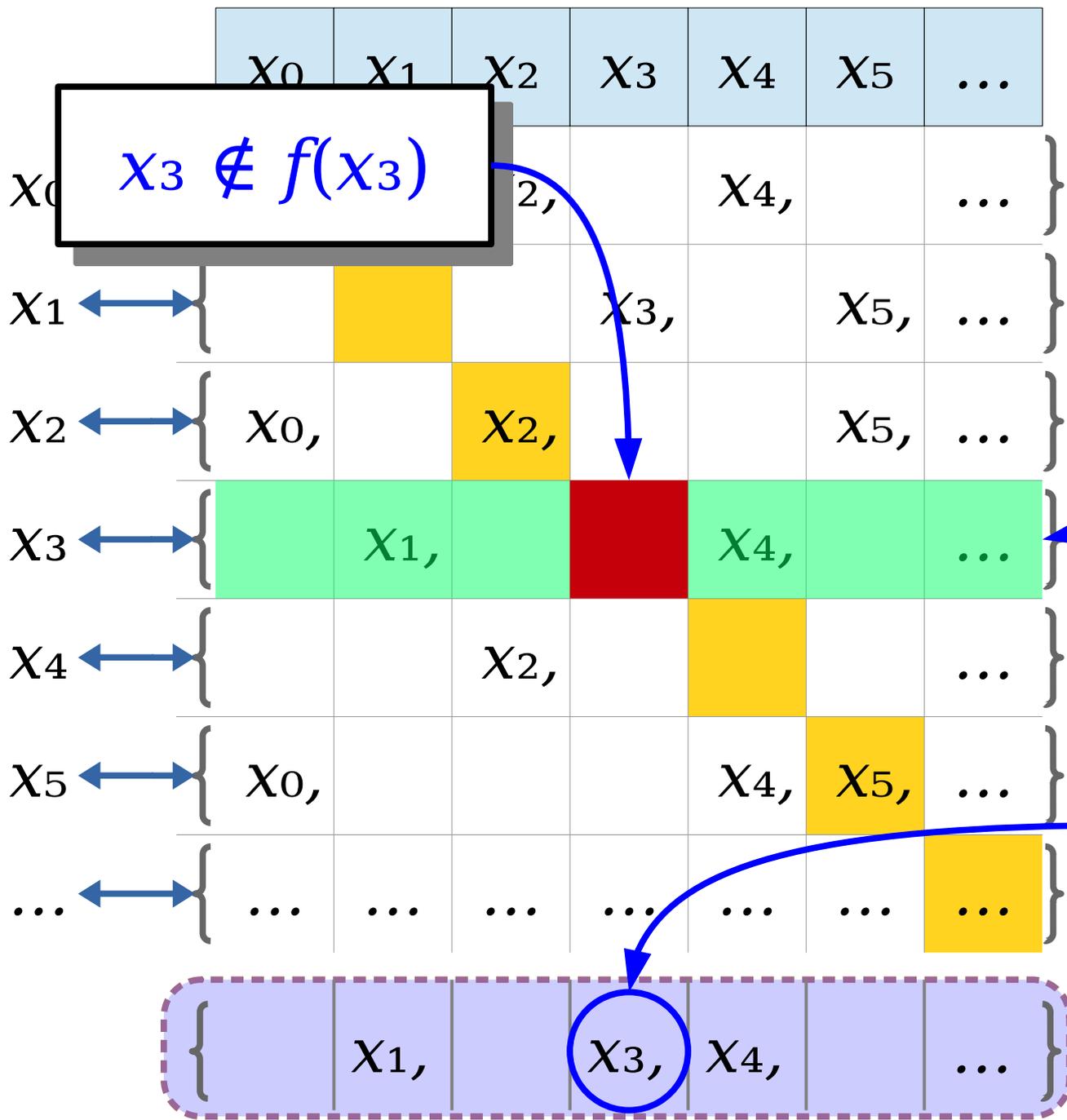
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$f(x_1)$



Why is x_1 in this set?

This is a drawing of our function $f: S \rightarrow \wp(S)$.

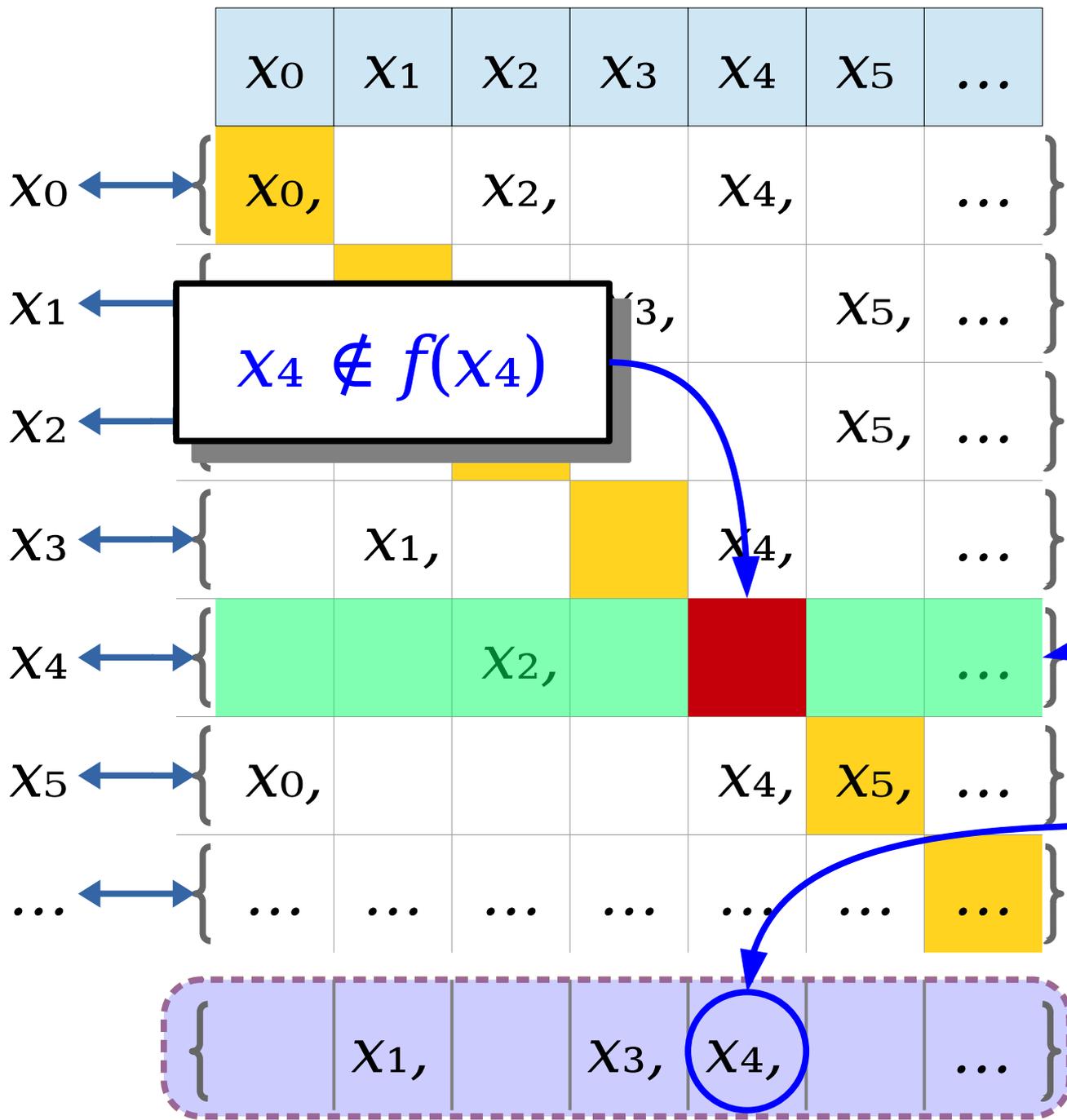


$x_3 \notin f(x_3)$

$f(x_3)$

Why is x_3 in this set?

This is a drawing of our function $f: S \rightarrow \wp(S)$.



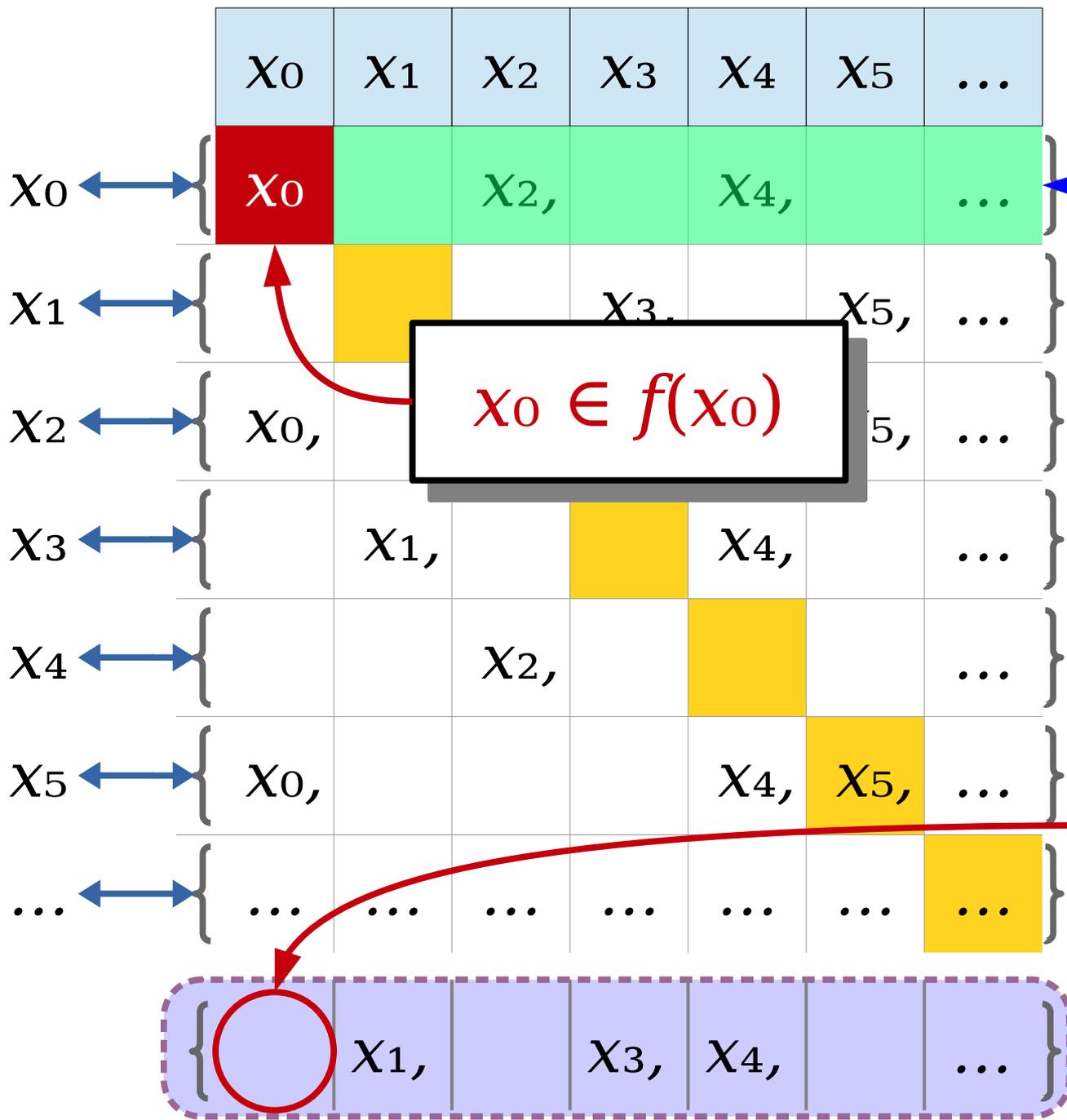
$x_4 \notin f(x_4)$

$f(x_4)$

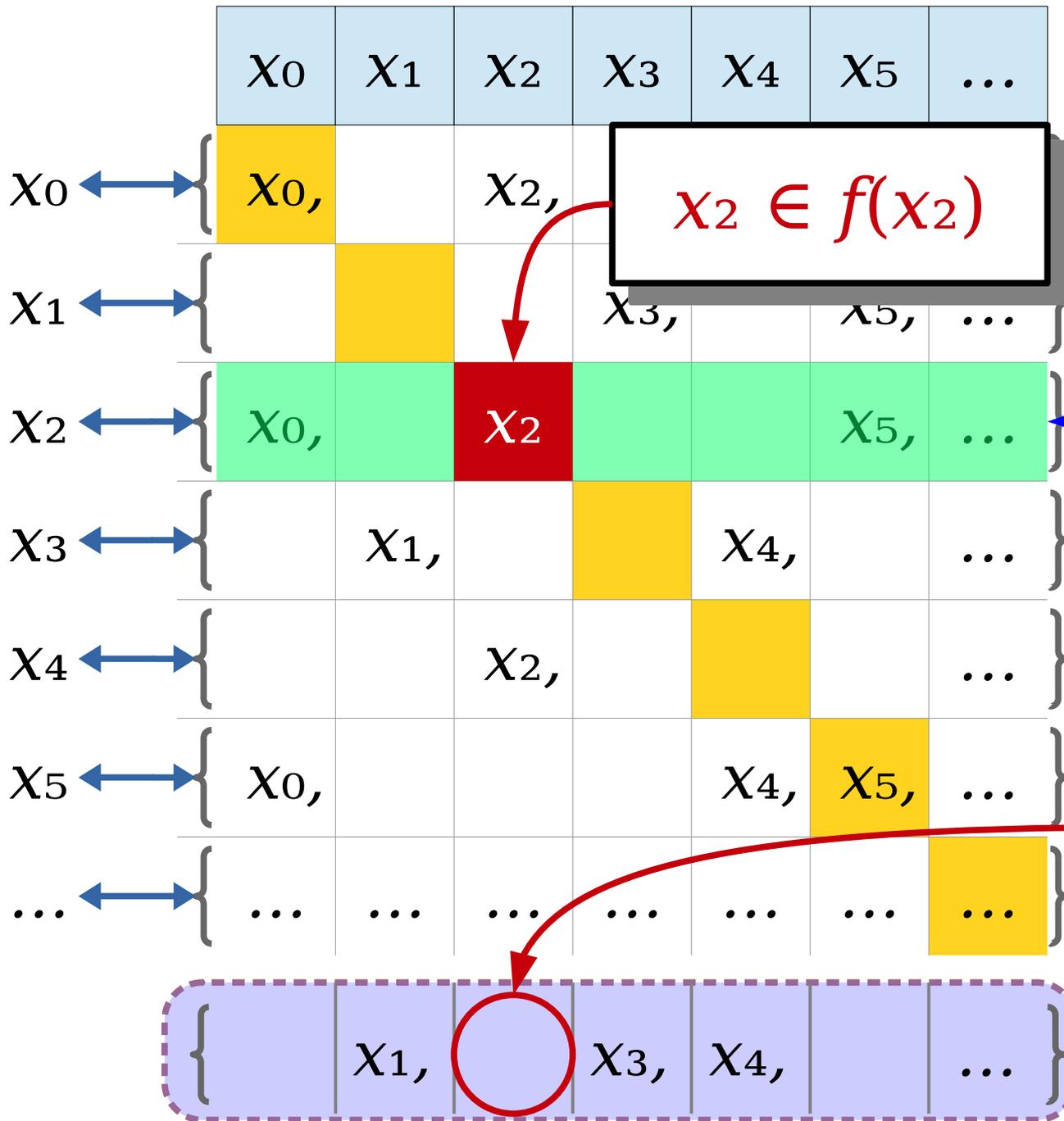
Why is x_4 in this set?

$x_4,$

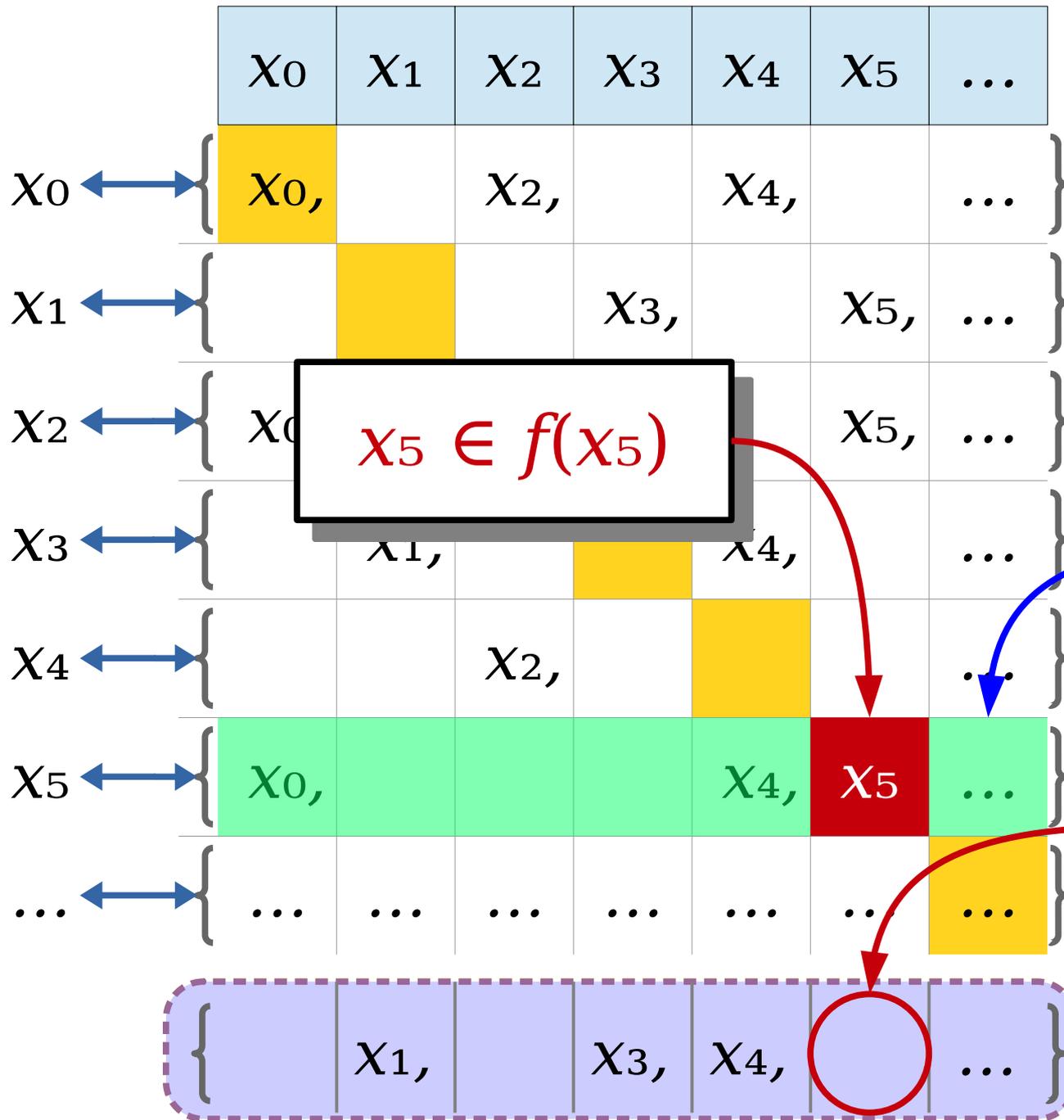
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	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 ↔	$x_0,$		$x_2,$		$x_4,$...
x_1 ↔				$x_3,$		$x_5,$...
x_2 ↔	$x_0,$		$x_2,$			$x_5,$...
x_3 ↔		$x_1,$			$x_4,$...
x_4 ↔			$x_2,$...
x_5 ↔	$x_0,$...
...

If $x \notin f(x)$, include x in the set.
 If $x \in f(x)$, exclude x from the set.

- { $x_1,$ $x_3,$ $x_4,$... }

This is a drawing of our function $f : S \rightarrow \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 \leftrightarrow	$x_0,$		$x_2,$		$x_4,$...
x_1 \leftrightarrow				$x_3,$		$x_5,$...
x_2 \leftrightarrow	$x_0,$		$x_2,$			$x_5,$...
x_3 \leftrightarrow		$x_1,$			$x_4,$...
x_4 \leftrightarrow			$x_2,$...
x_5 \leftrightarrow	$x_0,$...
...

If $x \notin f(x)$, include x in the set.
 If $x \in f(x)$, exclude x from the set.
 Define $D = \{ x \in S \mid x \notin f(x) \}$

- $\{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

The Diagonal Set

- For any set S and function $f : S \rightarrow \wp(S)$, we can define a set D as follows:

$$D = \{ x \in S \mid x \notin f(x) \}$$

(“The set of all elements x where x is not an element of the set $f(x)$.”)

- This is a formalization of the set we found in the previous picture.
- Using this choice of D , we can formally prove that no function $f : S \rightarrow \wp(S)$ is a bijection.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

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Proof:

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Starting with f , we define the set

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The Big Recap

- We define equal cardinality in terms of bijections between sets.
- Lots of different sets of infinite size have the same cardinality.
- Although the syntax of set cardinality makes it look like we're working with quantities, we are really working in terms of bijections between sets.
- Cantor's theorem can be formalized in terms of surjectivity.

Next Time

- ***Graphs***
 - A ubiquitous, expressive, and flexible abstraction!
- ***Properties of Graphs***
 - Building high-level structures out of lower-level ones!

Appendix: More Function Proofs

Proof 1: Composing surjections yields a surjection.

Theorem: If $f : A \rightarrow B$ is surjective and $g : B \rightarrow C$ is surjective, then $g \circ f : A \rightarrow C$ is also surjective.

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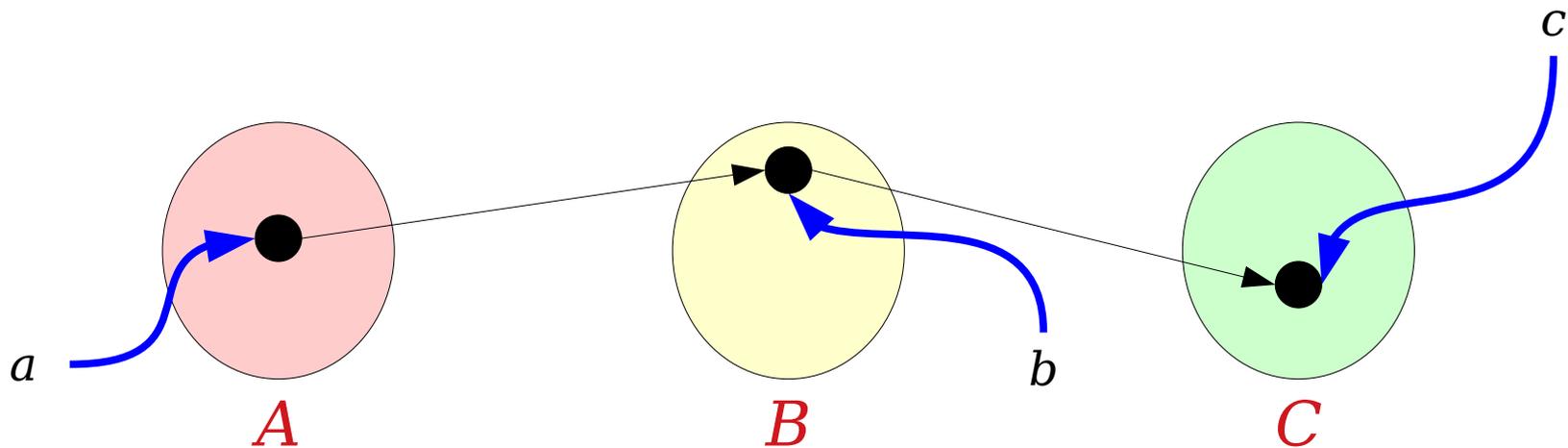
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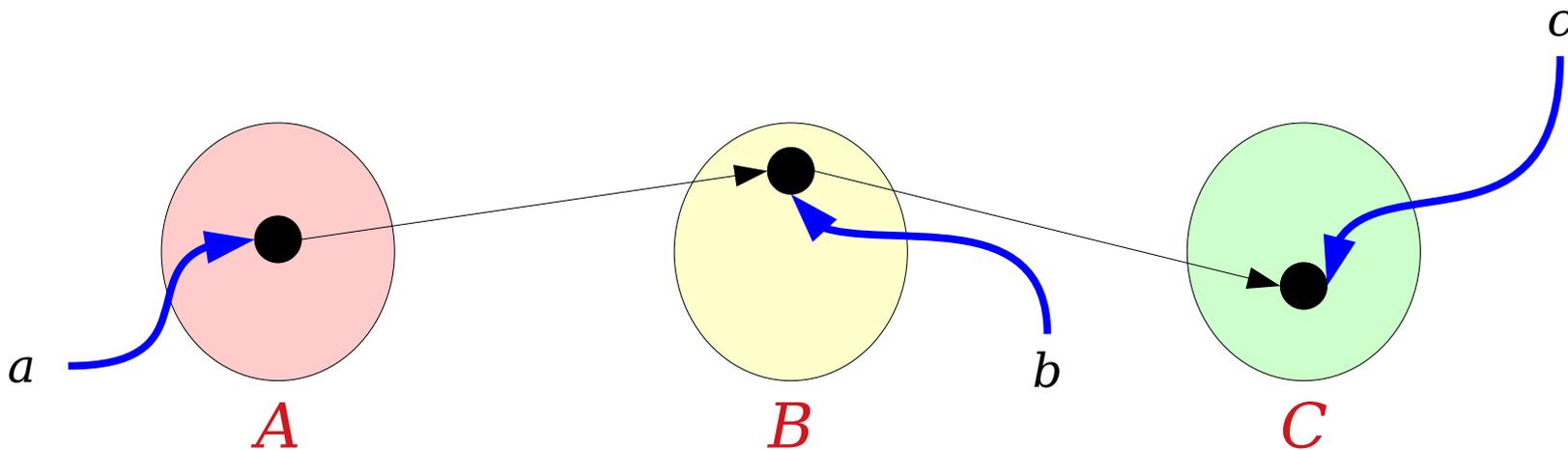
Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary surjections. We will prove that the function $g \circ f : A \rightarrow C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.



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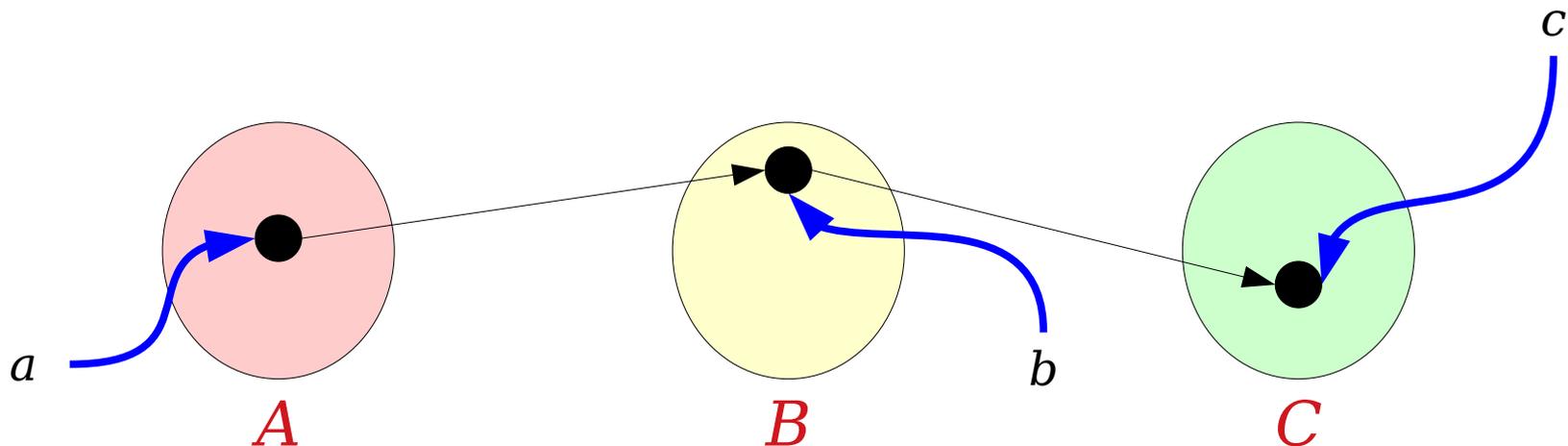
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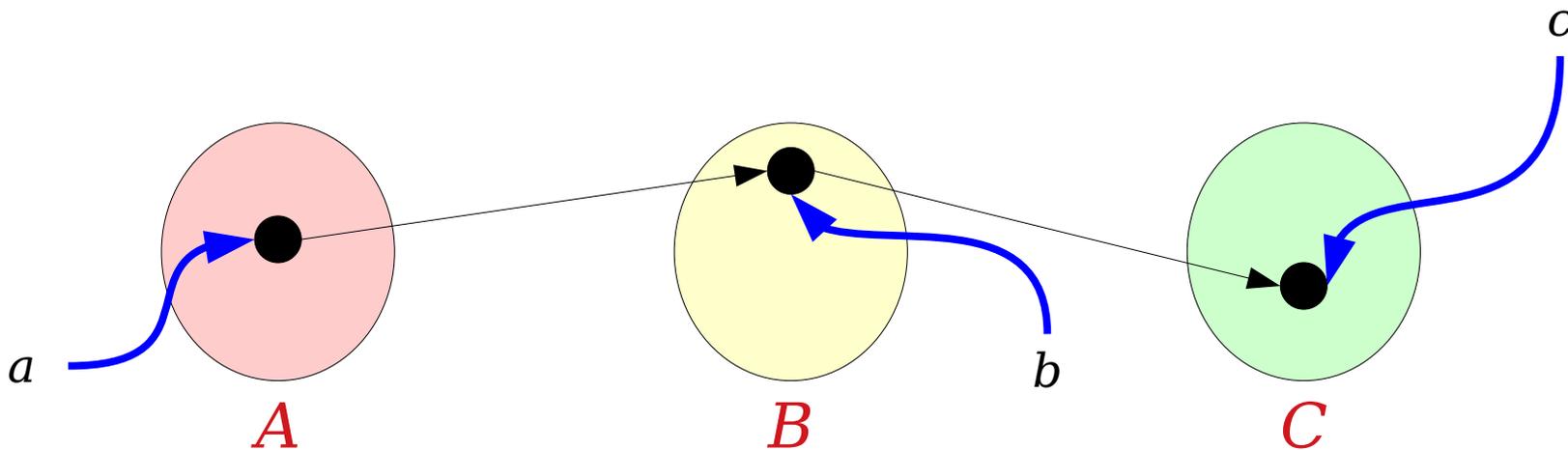
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Consider any $c \in C$. Since $g : B \rightarrow C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \rightarrow B$ is surjective, there is some $a \in A$ such that $f(a) = b$.



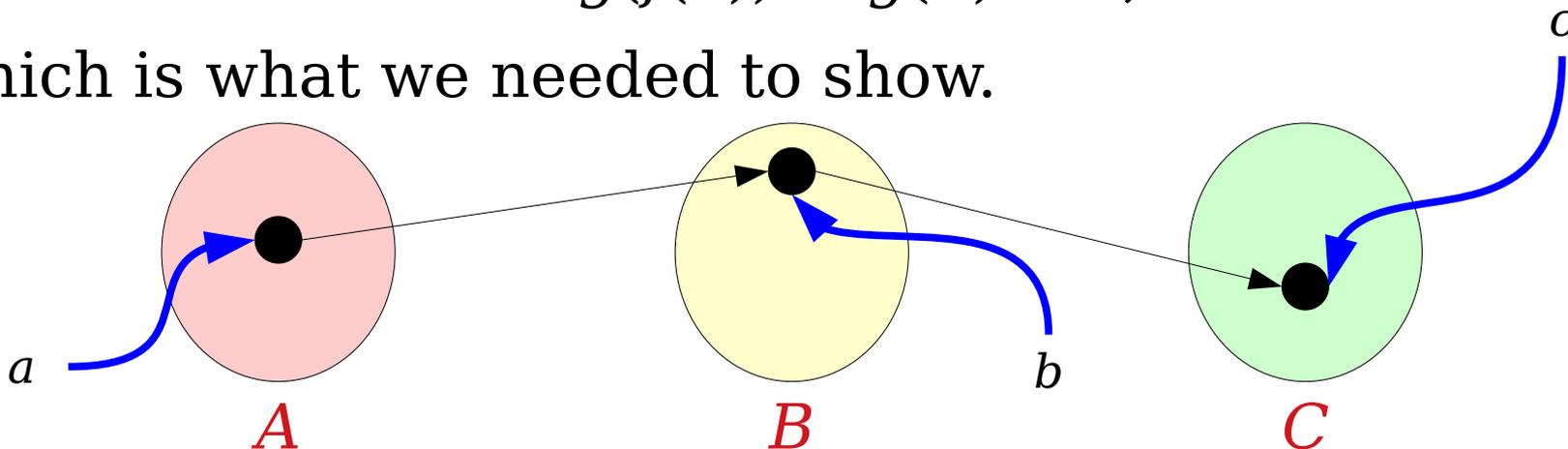
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Consider any $c \in C$. Since $g : B \rightarrow C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \rightarrow B$ is surjective, there is some $a \in A$ such that $f(a) = b$. Then we see that

$$g(f(a)) = g(b) = c,$$

which is what we needed to show.



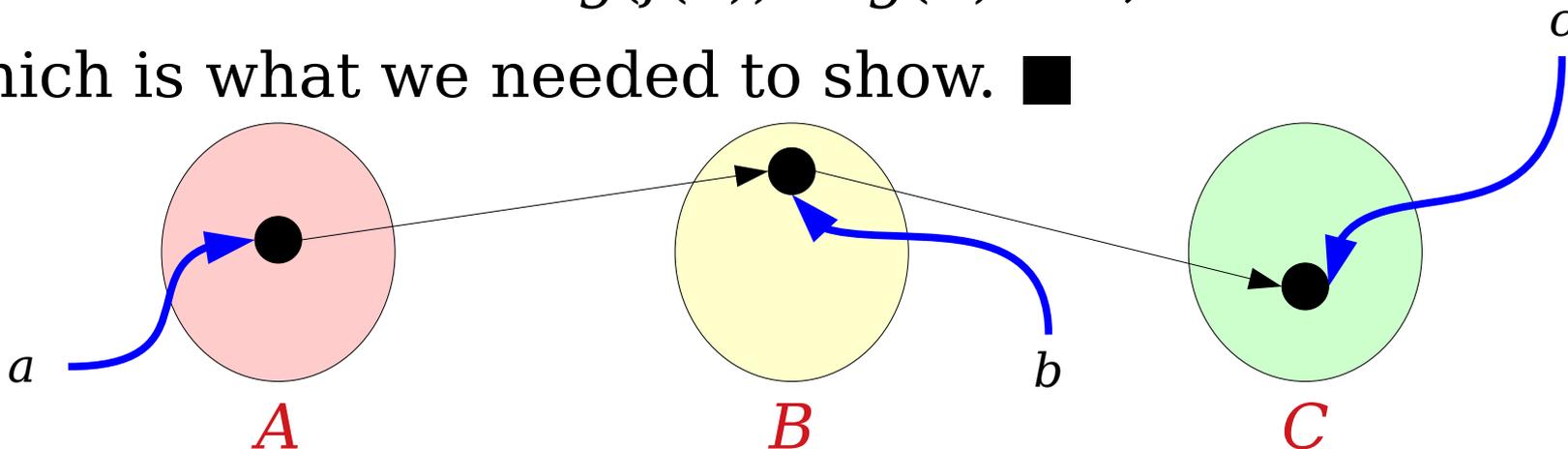
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Proof 2: $|A| = |A|.$

Theorem: For any set A , we have $|A| = |A|$.

Proof: Consider any set A , and let $f : A \rightarrow A$ be the function defined as $f(x) = x$. We will prove that f is a bijection.

First, we'll show that f is injective. Pick any $x_1, x_2 \in A$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. Since $f(x_1) = f(x_2)$, we see by definition of f that $x_1 = x_2$, as required.

Next, we'll show that f is surjective. Consider any $y \in A$. We will prove that there is some $x \in A$ where $f(x) = y$. Pick $x = y$. Then $x \in A$ (since $y \in A$) and $f(x) = x = y$, as required. ■

Proof 3: $|A| = |B|$ and $|B| = |C|$
means that $|A| = |C|$.

Theorem: If A , B , and C are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof: Consider any sets A , B , and C where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from A to C .

Since $|A| = |B|$, we know that there is some bijection $f : A \rightarrow B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \rightarrow C$.

Consider the function $g \circ f : A \rightarrow C$. Since g and f are bijections, their composition is a bijection. Thus $g \circ f$ is a bijection from A to C , so $|A| = |C|$, as required. ■

Proof 4: $|A| = |B|$ means $|B| = |A|$.

Inverse Functions

- If $f : A \rightarrow B$ is a function, the **inverse of f** , denoted f^{-1} , is a function $f^{-1} : B \rightarrow A$ where the following is true:

$$\forall a \in A. \forall b \in B. (f(a) = b \leftrightarrow f^{-1}(b) = a)$$

- **Theorem:** A function f has an inverse if and only if f is a bijection.
 - It's worth thinking about why this is! This isn't obvious, but it makes sense with some tinkering.
- **Theorem:** If f is a bijection, then so is f^{-1} .
 - Intuition: f^{-1} has f as its inverse, so the above theorem says that f^{-1} must be a bijection.
- These are great exercises if you're up for a challenge!

Theorem: If A and B are sets where $|A| = |B|$, then $|B| = |A|$.

Proof: Pick any sets A and B where $|A| = |B|$. We need to show that $|B| = |A|$.

Since $|A| = |B|$, we know that there is a bijection $f : A \rightarrow B$. Therefore, the function $f^{-1} : B \rightarrow A$ exists and is a bijection. Thus there is a bijection from B to A , so we conclude that $|B| = |A|$. ■